CSE 548: Algorithms

Amortized Analysis

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Amortization

The spreading out of capital expenses for intangible assets over a specific period of time (usually over the asset's useful life) for accounting and tax purposes.

- A clever trick used by accountants to average large one-time costs over time.
- In algorithms, we use amortization to spread out the cost of expensive operations.
    - Example: Re-sizing a hash table.

Topics

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Note: We are not making an "average case" argument about inputs. *We are still talking about worst-case performance.*

Summation Example: Binary Counter

- What is the worst-case runtime of *incr*?

Incr(B[0..])

$i = 0$

while $B[i] = 1$

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- Thus, amortized cost per *incr* is $O(1)$

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  - *Important: ensure you have charged enough.*
    - We have satisfied this: a bit can be flipped from 1 to 0 only once after it is flipped from 0 to 1.
- Now we ignore costs of 1 to 0 flips in the algorithm
  - There is only one 0-to-1 bit flipping per call of *incr*!
  - So, *incr* only costs 2 units for each invocation!

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- *Solution:*
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  - A pushed item can be popped only once, so we have charged enough
  - Now, ignore pop's altogther, and trivially arrive at $O(1)$ amortized cost for the sequence of push/pop operations!

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  - as the data structure changes and "releases" stored energy
- A more sophisticated technique that allows "charges" or "taxes" to be stored within nodes in a data structure and used subsequently at a later time.

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Counter:

- Define potential as the number one 1-bits
- Changing a 0 to 1 costs 2 units, one for the operation and one to pay for increase in potential
- Changes of 1 to 0 can now be paid by released potential.

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1. Try to guess the table size right; if you guessed wrong, put up with the pain of low performance.
2. Quit complaining, bite the bullet, and rehash as needed;
3. *Amortize*: Rehash as needed, *and* prove that it does not cost much!

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Hmmm. This is growing like n^2, so amortized cost will be $O(n)$

Need to try a different approach.

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The general recurrence is $T(n) = T(n/2) + 1.5n$, which is linear.

So, amortized cost is constant!

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Alternatively, we can think in terms of *charging*.

Each insert operation can be charged 3 units of cost:

- One for the insert operation
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Thus, rehashing

- increases the costs of insertions by a factor of 3.
- lookup costs are unchanged.

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- Each insert (after α reaches 0.5) costs 3 units: one for the operation, and 2 for the increase in potential.
- When α reaches 1, the potential is $2k$. After resizing to $2k$, potential falls to 0, and the released $2k$ cost pays for rehashing $2k$ elements.

Amortized Rehash (6)

- What if we increase the size by a factor less than 2?
  - Is there a threshold $t > 1$ such that expansion by a factor less than t won't yield amortized constant time?
- What happens if we want to support both deletes and inserts, and want to make sure that the table never uses more than k times the actual number of elements?
  - Is there a minimum value of k for which this can be achieved?
  - Do you need a different threshold for expansion and contraction? Are there any constraints on the relationship between these two thresholds to ensure amortized constant time?

Amortized performance of Vectors vs Lists

Linked lists: Data structures of choice if you don't know the # of elements in advance.

Space inefficient: 2x or more memory for very small objects.

Poor cache performance: Pointer chasing is cache unfriendly.

Sequential access: No fast access to kth element.

Vectors: Dynamically-sized arrays have none of these problems. But resizing is expensive.

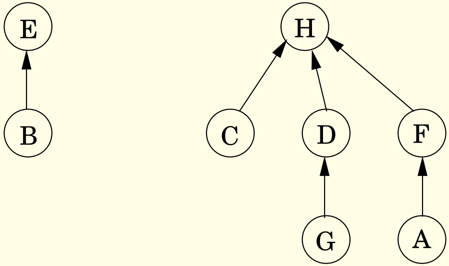
- Is it possible to achieve good amortized performance?
- When should the vector be expanded/contracted?
- What operations can we support in constant amortized time? Inserts? insert at end? concatenation?

Strings: We can raise similar questions as Vectors.

Disjoint Sets

- Represent disjoint sets as “inverted trees”
- Each element has a parent pointer π
- To compute the union of set A with B, simply make B’s root the parent of A’s root.

A directed-tree representation of two sets $\{B, E\}$ and $\{A, C, D, F, G, H\}$.



Disjoint Sets (2)

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procedure makeset(x)
π(x) = x
rank(x) = 0

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Amortized complexity

- Can you construct a worst-case example, where N operations take $O(N^2)$ time?
- Can we improve this?

Disjoint Sets with Union by Depth

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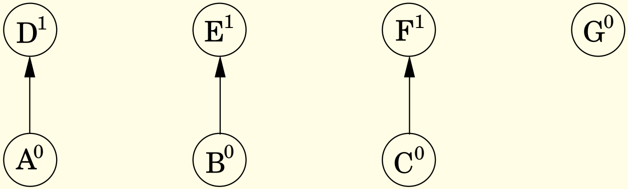
```
procedure union(x, y)
r_x = find(x)
r_y = find(y)
if r_x = r_y :   return
if rank(r_x) > rank(r_y):
    π(r_y) = r_x
else:
    π(r_x) = r_y
    if rank(r_x) = rank(r_y) :
        rank(r_y) = rank(r_y) + 1
```

Disjoint Sets with Union by Depth (2)

Figure 5.6 A sequence of disjoint-set operations. Superscripts denote rank.

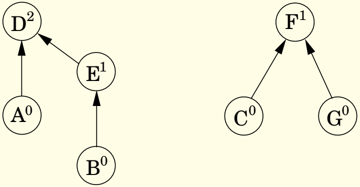
After makeset(A), makeset(B), . . . , makeset(G):

After union(A, D), union(B, E), union(C, F):

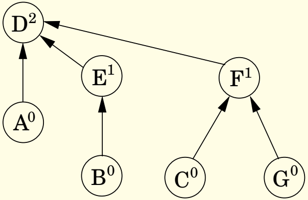


Disjoint Sets with Union by Depth (3)

After $\texttt{union}(C, G), \texttt{union}(E, A)$:



After $\texttt{union}(B, G)$:



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- Asymptotic complexity of `makeset` unchanged.
  - `union` has become a bit more expensive, but only modestly.
- What about `find`?
  - A sequence of N operations can create at most N elements
  - So, maximum set size is $O(N)$

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This requires find to be moved out of union into a separate operation, and hence the total number of operations increases, but only by a constant factor.

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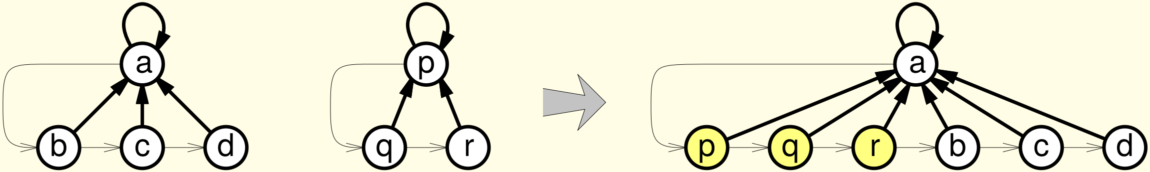

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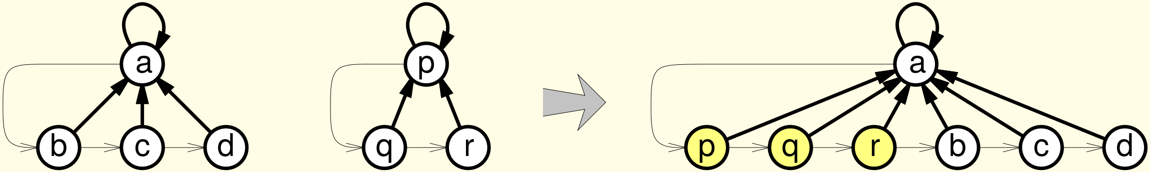

Problem: Worst-case complexity of `union` becomes $O(n)$

Solution:

- Merge smaller set with larger set
- Amortize cost of `union` over other operations

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- Thus, amortized cost of n operations, consisting of some mix of `makeset`, `find` and `union` is at most $n \log n$

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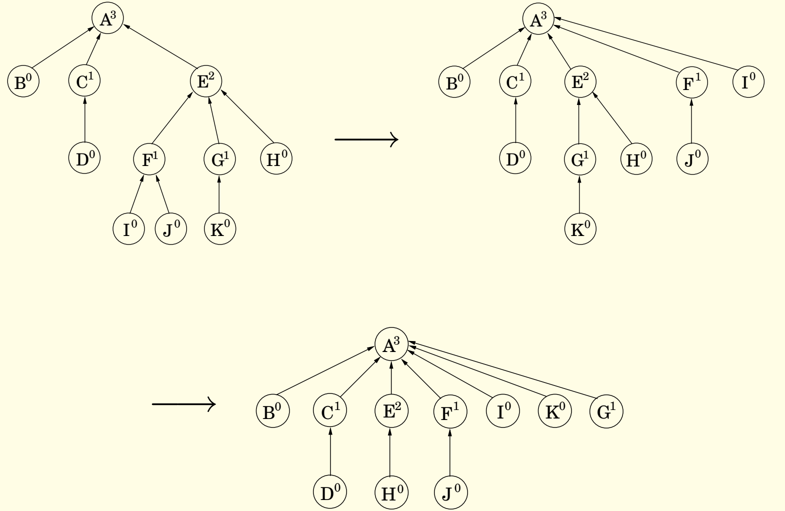
  - As a result, subsequent calls to find x or its parents become cheap.
- From here on, we let *rank* be defined by the *union* algorithm
  - For root node, *rank* is same as depth
  - But once a node becomes a non-root, its *rank* stays fixed,
    - *even when path compression decreases its depth.*

Disjoint sets w/ Path compression: Illustration

`find`(I) followed by `find`(K)



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Note that 2^{65536} is approximately a $20,000$ digit decimal number.

We will never be able to store input of that size, at least not in our universe. (Universe contains may be 10^{100} elementary particles.)

So, we might as well treat $\log^*(n)$ as $O(1)$.

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- We will spread this cost across all n operations, thus contributing $O(\log^* n)$ to each operation.

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- After $m + 1$th `find`, the `find` operation will pay for pointer updates, as $G(\pi(p)) > G(p)$ from here on.