CSE 548: Algorithms

Fall 2022

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Divide-and-Conquer: A versatile strategy

Steps

- Break a problem into subproblems that are smaller instances of the same problem
- Recursively solve these subproblems
- Combine these answers to obtain the solution to the problem

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Benefits

Conceptual simplification

Speed up:

- rapidly (exponentially) reduce problem space
- exploit commonalities in subproblem solutions

Parallelism: Divide-and-conquer algorithms are amenable to parallelization

Locality: Their depth-first nature increases locality, extremely important for today's processors.

Topics

1. Warmup
   - Overview
   - Search
   - Exponentiation
2. Sorting
   - Mergesort
   - Recurrences
   - Quicksort
   - Lower Bound
   - Radix sort
3. Selection
   - Select k-th min
   - Priority Queues
4. Multiplication
   - Matrix Multiplication
   - Integer multiplication

Binary Search

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B-Tree: Hybrid of the above two. Root stores an array M of m keys, and has $m + 1$ children. Use binary search on M to identify which child can contain k, recursively search that subtree.

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ExpBySquaring(n, x)

if $n > 1$
  $y = ExpBySquaring(\lfloor n/2 \rfloor, x^2)$
  if $odd(n)$ $y = x * y$
  return y
else return x

Merge Sort

function mergesort($a[1 \ldots n]$)
Input: An array of numbers $a[1 \ldots n]$
Output: A sorted version of this array

if $n > 1$:
  return merge(mergesort($a[1 \ldots \lfloor n/2 \rfloor]$), mergesort($a[\lfloor n/2 \rfloor + 1 \ldots n]$))
else:
  return a

Merge Sort (Continued)

```
function merge(x[1...k], y[1...l])
if k = 0:   return y[1...l]
if l = 0:   return x[1...k]
if x[1] ≤ y[1]:
   return x[1] ∘ merge(x[2...k], y[1...l])
else:
   return y[1] ∘ merge(x[1...k], y[2...l])
```

Merge Sort Illustration

$\boxed{10}$ $\boxed{2}$ $\boxed{5}$ $\boxed{3}$ $\boxed{7}$ $\boxed{13}$ $\boxed{1}$ $\boxed{6}$

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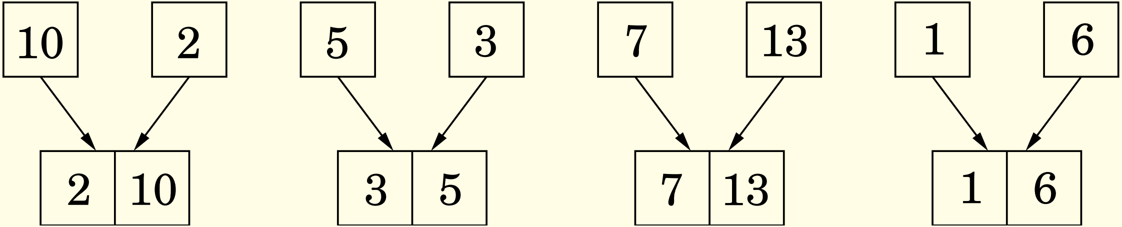

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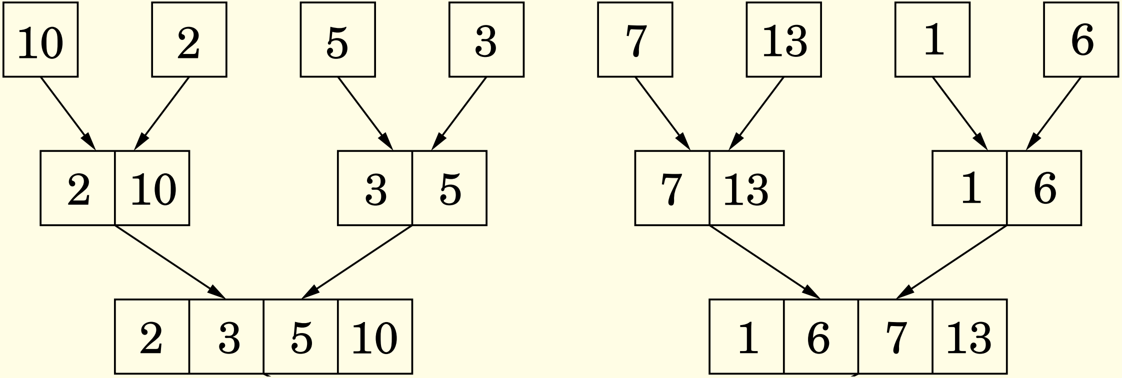

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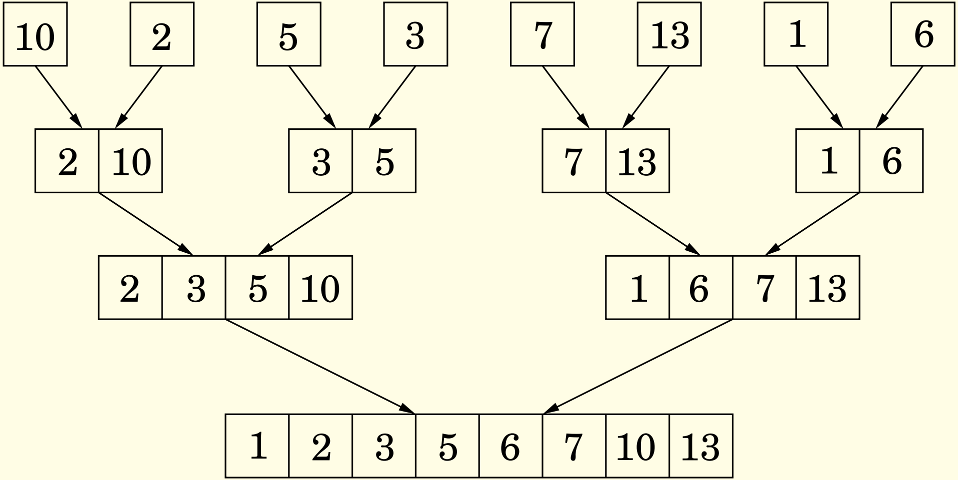

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- $\mathit{mergesort}(A)$ makes two recursive invocations of itself, each with an array half the size of A
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- In divide-and-conquer algorithms, we often encounter recurrences of the form

$$T(n) = aT\left(\frac{n}{b}\right) + O(n^d)$$

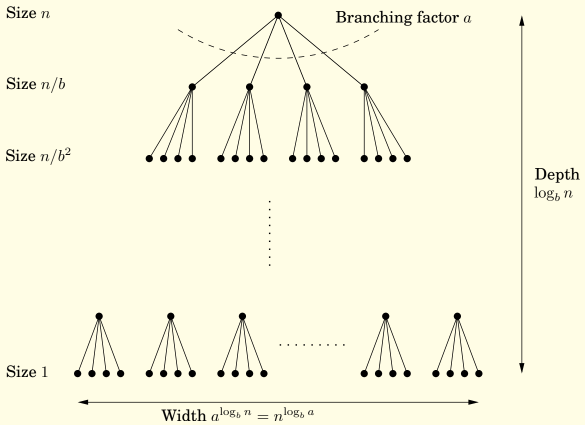
Can we solve them once for all?

Master Theorem

If $T(n) = aT\left(\frac{n}{b}\right) + O(n^d)$ for constants $a > 0$, $b > 1$, and $d \geq 0$, then

$$T(n) = \begin{cases} O(n^d), & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

Proof of Master Theorem



Can be proved by induction, or by summing up the series where each term represents the work done at one level of this tree.

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- Guess and check (prove by induction)
  - expand recursion for a few steps to make a guess
  - in principle, can be applied to any recurrence
- Akra-Bazzi method (not covered in class)
  - recurrences can be much more complex than that of Master theorem

Quicksort

$qs(A, l, h)$ /*sorts $A[l \ldots h]$*/

if $l >= h$ **return**;

$(h_1, l_2) =$

$partition(A, l, h)$;

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$partition(A, l, h)$

$k = selectPivot(A, l, h)$; $p = A[k]$;
$swap(A, h, k)$;
$i = l - 1$; $j = h$;
while *true* **do**
    do $i{+}{+}$ **while** $A[i] < p$;
    do $j{-}{-}$ **while** $A[j] > p$;
    if $i \geq j$ **break**;
    $swap(A, i, j)$;
$swap(A, i, h)$
return $(j, i + 1)$

Analysis of Runtime of *qs*

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Lucky/unlucky split: Alternate between best- and worst-case splits.

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T(n) &= n + T(1) + \boxed{\text{T(n-1)}} + n \quad \text{(worst case split)} \\
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Three-fourths split:

$$T(n) = n + T(0.25n) + T(0.75n) \leq n + 2T(0.75n) = O(n \log n)$$

Average case analysis of *qs*

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Set up the recurrence: When all permutations are qually likely, the selected pivot has an equal chance of ending up at the i^{th} position in the sorted order, for all $1 \le i \le n$. Thus, we have the following recurrence for the average case:

$$T(n) = n + \frac{1}{n} \sum_{i=1}^{n-1} (T(i) + T(n-i))$$

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Solve recurrence: Cannot apply the master theorem, but since it seems that we get an $O(n \log n)$ bound even in seemingly bad cases, we can try to establish a $cn \log n$ bound via induction.

Establishing average case of *qs*

- Establish base case. (Trivial.)
- Induction step involves summation of the form $\sum_{i=1}^{n-1} i \log i$.

  Attempt 1: Bound $\log i$ above by $\log n$. (Induction fails.)

  Attempt 2: Split the sum into two parts:

$$\sum_{i=1}^{n/2} i \log i + \sum_{i=n/2+1}^{n-1} i \log i$$

  and apply the approximation to each half. (Succeeds with $c \geq 4$.)

  Attempt 3: Replace summation with integration. (See "Integration method" in Summations.)

$$\int_{x=1}^{n} x \log x = \frac{x^2}{2}\left(\log x - \frac{1}{2}\right)\Big|_{x=1}^{n}$$

  (Succeeds with the constraint $c \geq 2$.)

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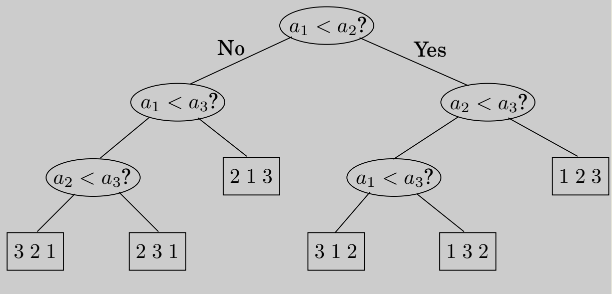
  - For randomized algorithms, we talk about *expected complexity*, which is an average over all possible values of the random variable.
- If pivot index is picked uniformly at random over the interval $[l, h]$, then:
  - every array element is equally likely to be selected as the pivot
  - every partition is equally likely
  - thus, *expected* complexity of *randomized* quicksort is given by the same recurrence as the *average* case of qs.

Lower bounds for comparison-based sorting

- Sorting algorithms can be depicted as trees: each leaf identifies the input permutation that yields a sorted order.



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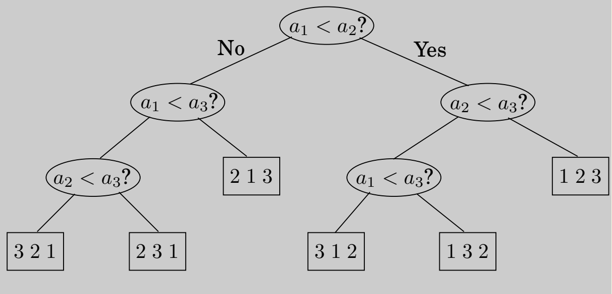

- The tree has $n!$ leaves, and hence a height of $\log n!$. By Stirling's approximation, $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, so, $\log n! = O(n \log n)$

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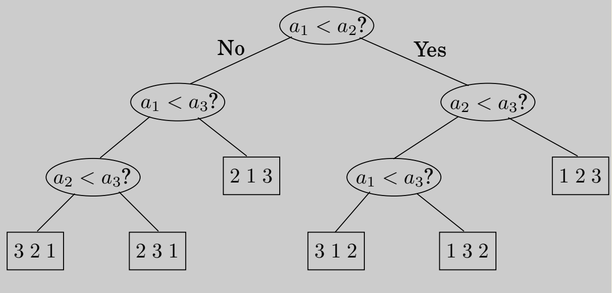

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- No *comparison-based* sorting algorithm can do better!

Bucket sort

Overview

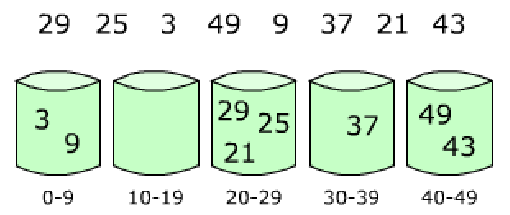
Divide: Partition input into intervals (buckets), based on key values

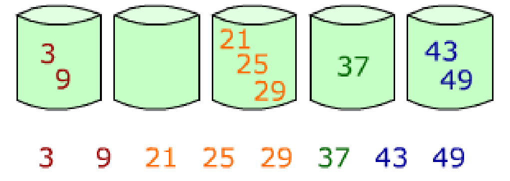
- Linear scan of input, drop into appropriate bucket

Recurse: Sort each bucket

Combine: Concatenate bin contents

Example





Bucket sort (Continued)

- Bucket sort generalizes quicksort to multiple partitions
  - Combination = concatenation
  - Worst case quadratic bound applies
  - But performance can be much better if input distribution is uniform.
    Exercise: What is the runtime in this case?
- Used by letter sorting machines in post offices

Counting Sort

Special case of bucket sort where each bin corresponds to an interval of size 1.

- No need to recurse. Divide $=$ conquered!
- Makes sense only if range of key values is small (usually constant)
- Thus, counting sort can be done in $O(n)$ time!

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  - *Hmm. How did we beat the $O(n \log n)$ lower bound?*

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- *Note:* Radix sort does not divide inputs into smaller subsets

  If you think of input as multi-dimensional data, then we break down the problem to each dimension.

Stable sorting algorithms

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- Merge sort and LSD sort are stable. Quicksort is not stable.

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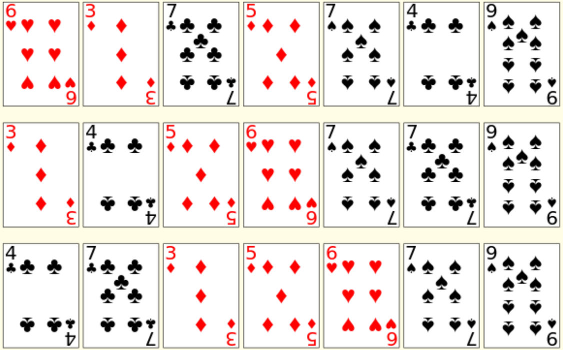

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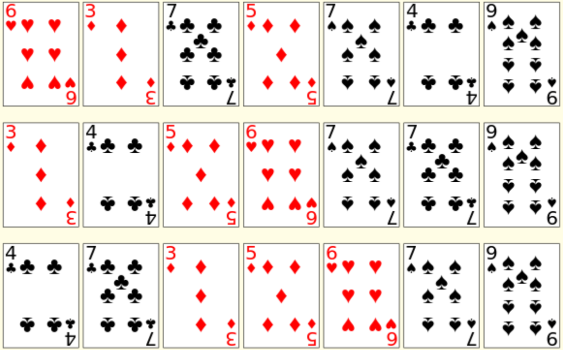

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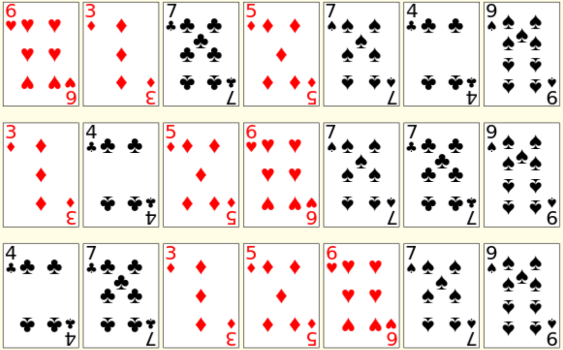
- Other examples: sorting spread sheets or tables on web pages



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- Easy to devise an $O(nl)$ algorithm, where n is the number of strings and l is the maximum size of any string.
  - But such an algorithm is *not* linear in input size.
- *Exercise:* Devise a linear-time string algorithm.

  Given a set $\mathcal{S}$ of strings, your algorithm should sort in $O(|\mathcal{S}|)$ time, where

$$|\mathcal{S}| = \sum_{s \in \mathcal{S}} |s|$$

Select k^{th} largest element

Obvious approach: Sort, pick k^{th} element — wasteful, $O(n \log n)$

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Complexity

Best case: Splits are even: $T(n) = n + T(n/2)$, which has an $O(n)$ solution.

Skewed 10%/90% $T(n) \leq n + T(0.9n)$ — still linear

Worst case: $T(n) = n + T(n-1)$ — *quadratic!*

Worst-case $O(n)$ Selection

Intuition: Spend a bit more time to select a pivot that ensures reasonably balanced partitions

MoM Algorithm [Blum, Floyd, Pratt, Rivest and Tarjan 1973]

Time Bounds for Selection

by .

Manuel Blum, Robert W. Floyd, Vaughan Pratt,
Ronald L. Rivest, and Robert E. Tarjan

Abstract

The number of comparisons required to select the i-th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm -- PICK. Specifically, no more than 5.4305 n comparisons are ever required. This bound is improved for

$O(n)$ Selection: MoM Algorithm

- Quick select (*qsel*) takes no time to pick a pivot, but then spends $O(n)$ to partition.
- Can we spend more time upfront to make a better selection of the pivot, so that we can avoid highly skewed splits?

Key Idea

- Use the selection algorithm itself to choose the pivot.
  - Divide into sets of 5 elements
  - Compute median of each set ($O(5)$, i.e., constant time)
  - Use selection recursively on these $n/5$ elements to pick their median
    - i.e., choose the median of medians (MoM) as the pivot
- Partition using MoM, and recurse to find kth largest element.

$O(n)$ Selection: MoM Algorithm

Theorem: MoM-based split won't be worse than 30%/70%

Result: Guaranteed linear-time algorithm!

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Result: Guaranteed linear-time algorithm!

Caveat: The constant factor is non-negligible; use as fall-back if random selection repeatedly yields unbalanced splits.

Selecting maximum element: Priority Queues

Heap

- A tree-based data structure for priority queues

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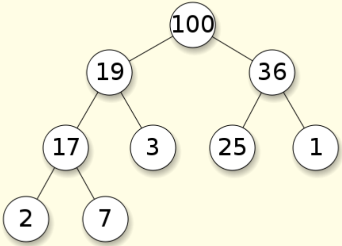
Heap property: F every subtree h of H

$$\forall k \in keys(h) \quad root(h) \geq k$$

where $keys(h)$ includes all keys within h



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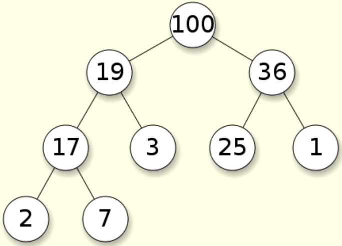

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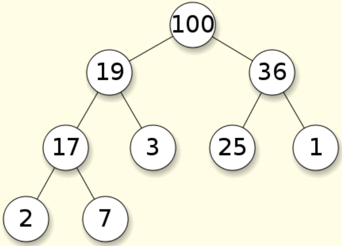
- Supports *insert*, *deleteMax* and *max*.



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Selecting maximum element: Priority Queues

Heap

- A tree-based data structure for priority queues

  Heap property: F every subtree h of H

  $$\forall k \in keys(h) \quad root(h) \geq k$$

  where $keys(h)$ includes all keys within h

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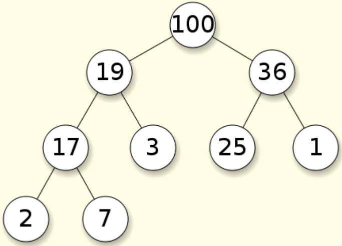

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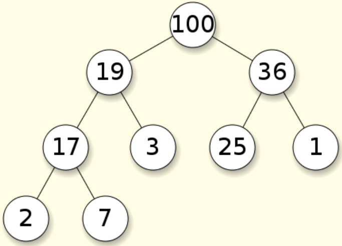

Task of maintaining max is distributed to subsets of the entire set; alternatively, it can be thought of as maintaining several parallel queues with a single head.

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Binary heap

Array representation: Store heap elements in breadth-first order in the array. Node i's children are at indices $2 * i$ and $2 * i + 1$

- Conceptually, we are dealing with a balanced binary tree

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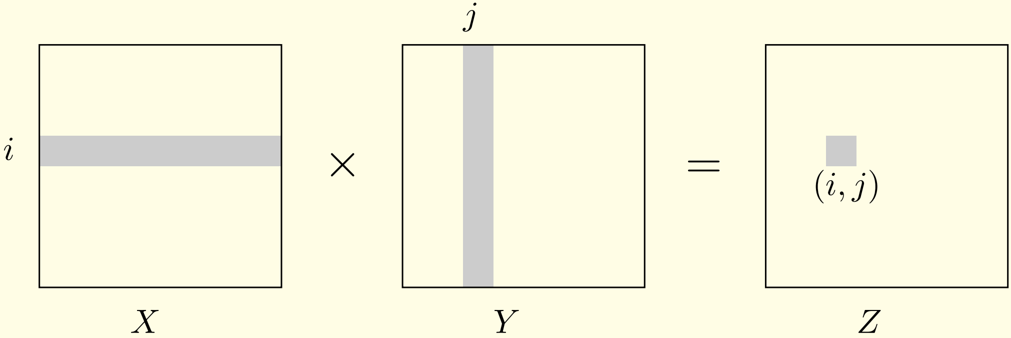
Heapsort: $O(n \log n)$ algorithm, *MkHeap* followed by n calls to *DeleteMax*

Matrix Multiplication

The product Z of two $n \times n$ matrices X and Y is given by

$$Z_{ij} = \sum_{k=1}^{n} X_{ik} Y_{kj} \quad \text{— leads to an } O(n^3) \text{ algorithm.}$$



Divide-and-conquer Matrix Multiplication

Divide X and Y into four $n/2 \times n/2$ submatrices

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ and } Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

Recursively invoke matrix multiplication on these submatrices:

$$XY = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

Divided, but did not conquer! $T(n) = 8T(n/2) + O(n^2)$, which is still $O(n^3)$

Strassen's Matrix Multiplication

Strassen showed that 7 multiplications are enough:

$$XY = \begin{bmatrix} P_6 + P_5 + P_4 - P_2 & P_1 + P_2 \\ P_3 + P_4 & P_1 - P_3 + P_5 - P_7 \end{bmatrix} \quad \text{where}$$

$$\begin{aligned} P_1 &= A(F - H) & P_5 &= (A + D)(E + H) \\ P_2 &= (A + B)H & P_6 &= (B - D)(G + H) \\ P_3 &= (C + D)E & P_7 &= (A - C)(E + F) \\ P_4 &= D(G - E) \end{aligned}$$

Now, the recurrence $T(n) = 7T(n/2) + O(n^2)$ has $O(n^{\log_2 7} = n^{2.81})$ solution!

Best-to-date complexity is about $O(n^{2.4})$, but this algorithm is not very practical.

Karatsuba's Algorithm

Same high-level strategy as Strassen — but predates Strassen.

Divide: n-digit numbers into halves, each with $n/2$-digits:

$$\begin{aligned} a &= \boxed{a_1}\boxed{a_0} = 2^{n/2}a_1 + a_0 \\ b &= \boxed{b_1}\boxed{b_0} = 2^{n/2}b_1 + b_0 \\ ab &= 2^n a_1 b_1 + 2^{n/2}(a_1 b_0 + b_1 a_0) + a_0 b_0 \end{aligned}$$

Key point — Instead of 4 multiplications, we can get by with 3 since:

$$a_1 b_0 + b_1 a_0 = (a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0$$

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Recursively compute $a_1 b_1, a_0 b_0$ and $(a_1 + a_0)(b_1 + b_0)$.

Recurrence $T(n) = 3T(n/2) + O(n)$ has an $O(n^{\log_2 3} = n^{1.59})$ solution!

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Note: This trick for using 3 (not 4) multiplications first noted by Gauss (1777-1855).

Faster algorithms for Integer Multiplication

- Toom-Cook Multiplication: Generalize Karatsuba
  - Divide into $n > 2$ parts
- FFT (Fast Fourier Transformation) based multiplication (Schonhage-Strassen)
- Can be more easily understood when integer multiplication is viewed as a polynomial multiplication.

Integer Multiplication Revisited

- An integer represented using digits

$$a_{n-1}\ldots a_0$$

over a base d (i.e., $0 \leq a_i < d$) is very similar to the polynomial

$$A(x) = \sum_{i=0}^{n-1} a_i x^i$$

Specifically, the value of the integer is $A(d)$.

- Integer multiplication follows the same steps as polynomial multiplication:

$$a_{n-1}\ldots a_0 \times b_{n-1}\ldots b_0 = (A(x) \times B(x))(d)$$

Polynomials: Basic Properties

Horner's rule

An n^{th} degree polynomial $\sum_{i=0}^{n} a_i x^i$ can be evaluated in $O(n)$ time:

$$((\cdots((a_n x + a_{n-1})x + a_{n-2})x + \cdots + a_1)x + a_0)$$

Roots and Interpolation

- An n^{th} degree polynomial $A(x)$ has exactly n roots $r_1, \ldots, r_n$. In general, r_i's are complex and need not be distinct.
- It can be represented as a product of sums using these roots:

$$A(x) = \sum_{i=1}^{n} a_i x^i = \prod_{i=0}^{n} (x_i - r_i)$$

- Alternatively, $A(x)$ can be specified uniquely by specifying $n+1$ points (x_i, y_i) on it, i.e., $A(x_i) = y_i$.

Operations on Polynomials

Representation	Add	Mult
Coefficients	$O(n)$	$O(n^2)$
Roots	?	$O(n)$
Points	$O(n)$	$O(n)$

Note: Point representation is the best for computation! But usually, only the coefficients are given

Solution: Convert to point form by *evaluating* $A(x)$ at selected points.

But conversion defeats the purpose: requires $O(n)$ evaluations, each taking $O(n)$ time, thus we are back to $O(n^2)$ total time.

Toom (and FFT) Idea: Choose evaluation points judiciously to speed up evaluation

Integer Multiplication Summary

- Algorithms implemented in libraries for arbitrary precision arithmetic, with applications in public key cryptography, computer algebra systems, etc.
- GNU MP is a popular library, uses various algorithms based on input size: naive, Karatsuba, Toom-3, Toom-4, or Schonhage-Strassen (at about 50K digits).
- Karatsuba is Toom-2. Toom-N is based on
  - Evaluating a polynomial at $2N$ points,
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