CSE 548: Algorithms

Greedy Algorithms

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  - Multiple solutions exist; pick one of low (or least) cost
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- So, use with a great deal of care
  - *Always need to prove optimality*
- If it is unpredictable, why use it?
  - *It simplifies the task!*

Making change

Given coins of denominations 25¢, 10¢, 5¢ and 1¢, make change for x cents ($0 < x < 100$) using *minimum number of coins.*

Greedy solution

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if ($x = 0$) **return**

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- Show that it is optimal
- Is it optimal for arbitrary denominations?

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Implies that a greedy algorithm can invoke itself recursively after making a greedy choice.

Knapsack Problem

- A sack that can hold a maximum of x lbs
- You have a choice of items you can pack in the sack
- Maximize the combined "value" of items in the sack

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Will a greedy algorithm work, with $x = 5$?

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Then the optimal solution for the problem includes the optimal choice of how to fill a knapsack of size y with the remaining items.

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Does not work for 0-1 knapsack because greedy choice property does not hold.

0-1 knapsack is NP-hard, but a pseudo-polynomial algorithm is available.

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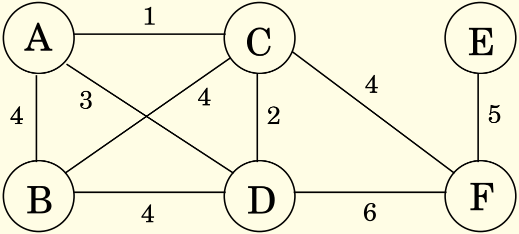
- A maximal subset of E such that the subgraph has no cycles
- A subset of E with $|V| - 1$ edges such that the subgraph is connected
- A subset of E such that there is a unique path between any two vertices in the subgraph

Minimal Spanning Tree (MST)

A spanning tree with *minimal cost.* Formally:

Input: An undirected graph $G = (V, E)$, a cost function $w : E \to \mathbb{R}$.

Output: A tree $T = (V, E')$ such that $E' \subseteq E$ that minimizes $\sum_{e \in E'} w(e)$



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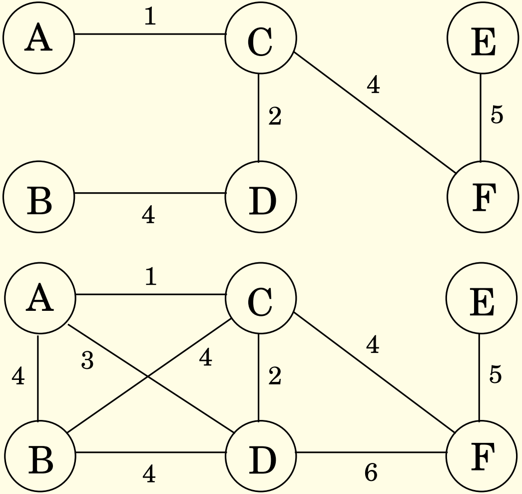
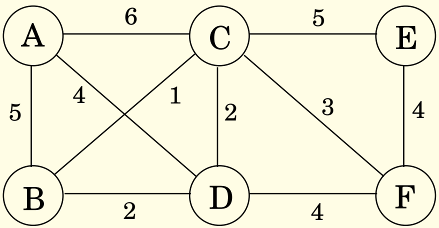
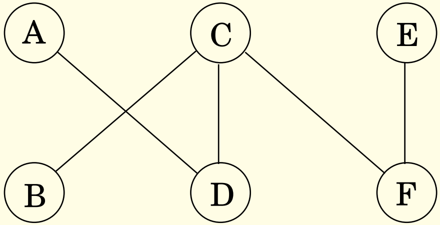


Kruskal's algorithm

- Start with the empty set of edges
- Repeat: add lightest edge that doesn't create a cycle

Adds edges B—C, C—D, C—F, A—D, E—F





Kruskal's algorithm

MST(V, E, w)

$X = \phi$

$Q = priorityQueue(E)$ // from min to max weight

while Q is nonempty

$e = deleteMin(Q)$

if e connects two disconnected components in (V, X) $X = X \cup \{e\}$

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  - Since neither e nor e' are in X, and Kruskal's chose e, $w(e') \geq w(e)$.
  - Replace e' by e in T to get another spanning tree T'. Either $w(T') < w(T)$, a contradiction to the assumption T is minimal; or $w(T') = w(T)$, and we have another MST T' consistent with $X \cup \{e\}$. In both cases, we have completed the induction step. ■

Kruskal's: Runtime complexity

$MST(V, E, w)$

$X = \phi$

$Q = priorityQueue(E, w)$ // from min to max weight

while Q is nonempty

  $e = deleteMin(Q)$

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- Priority queue: $O(\log |E|) = O(\log V)$ per operation
- Connectivity test: $O(\log V)$ per check using a disjoint set data structure

Thus, for $|E|$ iterations, we have a runtime of $O(|E| \log |V|)$

MST: Applications

Network design: Communication networks, transportation networks, electrical grid, oil/water pipelines, ...

Clustering: Application of minimum spanning forest (stop when $|X| = |V| - k$ to get k clusters)

Broadcasting: Spanning tree protocol in Ethernets

Information Theory and Coding

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- "surprise factor" — low probability event conveys more information; an event that is almost always likely ($p \approx 1$) conveys no information.
- Information content adds up: for two events e_1 and e_2, their combined information content is $-(\log p_1 + \log p_2)$

Information theory: Entropy

Information entropy

For a discrete random variable X that can take a value x_i with probability p_i, its entropy is defined as the *expectation* ("weighted average") over the information content of x_i:

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- Entropy is a measure of uncertainty
- Plays a fundamental role in many areas, including coding theory and machine learning.

Optimal code length

Shannon's source coding theorem

A random variable X denoting chars in an alphabet $\Sigma = \{x_1, \ldots, x_n\}$

- cannot be encoded in fewer than $H(X)$ bits.
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- Huffman coding: an algorithm that achieves this bound

Variable-length encoding

Let $\Sigma = \{A, B, C, D\}$ with probabilities $0.55, 0.02, 0.15, 0.28$.

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  - Let us use as many bits as the *information content* of a character
  - A uses 1 bit, B uses 6 bits, C uses 3 bits, and D uses 2 bits.
  - You get an average saving of 15%

    $0.55 * 1 + 0.02 * 6 + 0.15 * 3 + 0.28 * 2 = 1.68$ bits

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  - Lower bound (entropy)

    $$-(.5 \log_2 .5 + .02 \log_2 .02 + .14 \log_2 .14 + .27 \log_2 .27) = 1.51 \text{ bits}$$

Variable-length encoding

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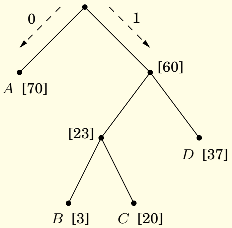
- Let us try fixing the codes, not just their lengths:

  $A = 0, D = 11, C = 101, B = 100.$

- Note: enough to assign 3 bits to B, not 6. So, average coding size reduces to 1.62.



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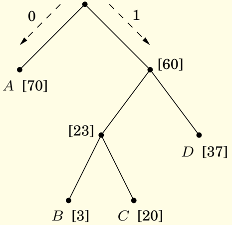
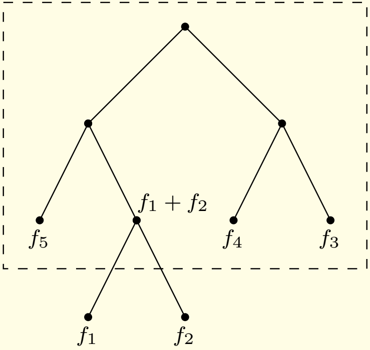


Prefix encoding

- No code is a prefix of another.
- Necessary property to enable decoding.
- Every such encoding can be represented using a full binary tree (either 0 or 2 children for every node)

Huffman encoding

- Build the prefix tree bottom-up
- Start with a node whose children are codewords c_1 and c_2 that occur least often
- Remove c_1 and c_2 from alphabet, replace with c' that occurs with frequency $f_1 + f_2$
- Recurse



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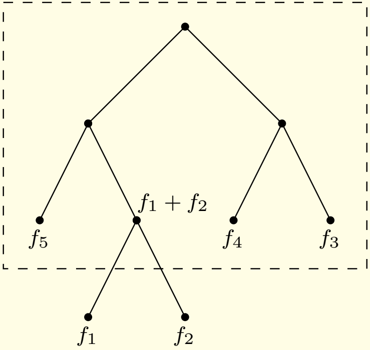
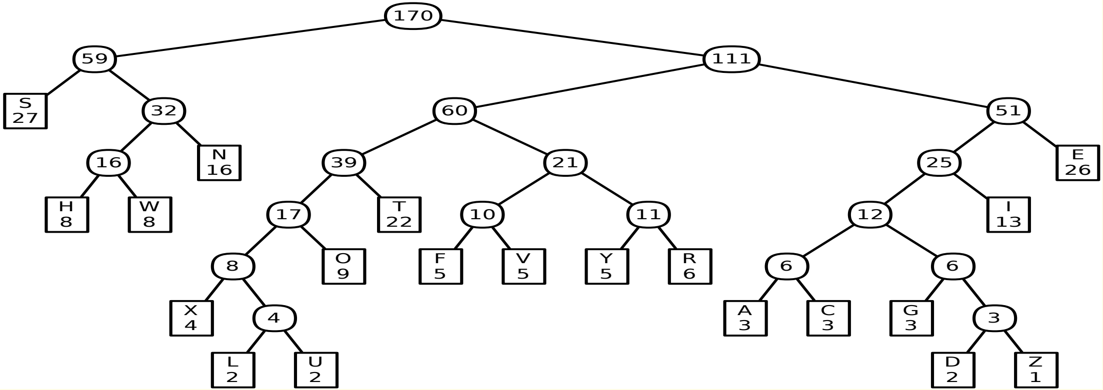


- How to make this algorithm fast?
- What is its complexity?

Huffman encoding: Example



This sentence contains three a's, three c's, two d's, twenty-six e's, five f's, three g's, eight h's, thirteen i's, two l's, sixteen n's, nine o's, six r's, twenty-seven s's, twenty-two t's, two u's, five v's, eight w's, four x's, five y's, and only one z. Images from Jeff Erickson's "Algorithms"

Uses about 650 bits, vs 850 for fixed-length (5-bit) code.

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  - Same argument holds for c_2

Huffman Coding: Applications

- Document compression
- Signal encoding
- As part of other compression algorithms (MP3, gzip, PKZIP, JPEG, ...)

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- Can the codebook be constructed on-the-fly?
  - Lempel-Ziv compression algorithms (gzip)

gzip Algorithm [Lempel-Ziv 1977]

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Encoding:

- Strings previously seen in the window are replaced by the pair (*offset*, *length*)
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Decoding: Interpret (*offset*, *length*) using the same window of W-bytes of preceding text. (Much faster than encoding.)

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- Examples
  - MST and clustering
  - Shortest path
  - Huffman encoding