CSE 548: Algorithms

Randomized Algorithms

R. Sekar

Example 1: Routing

- What is the best way to route a packet from X to Y, esp. in high speed, high volume networks

  A: Pick the shortest path from X to Y

  B: Send the packet to a random node Z, and let Z route it to Y (possibly using a shortest path from Z to Y)

Example 1: Routing

- What is the best way to route a packet from X to Y, esp. in high speed, high volume networks

  A: Pick the shortest path from X to Y

  B: Send the packet to a random node Z, and let Z route it to Y (possibly using a shortest path from Z to Y)

- Valiant showed in 1981 that surprisingly, B works better!
  - Turing award recipient in 2010

Example 2: Transmitting on shared network

- What is the best way for n hosts to share a common a network?
  - A: Give each host a turn to transmit
  - B: Maintain a queue of hosts that have something to transmit, and use a FIFO algorithm to grant access
  - C: Let every one try to transmit. If there is contention, use random choice to resove it.
- Which choice is better?

Topics

Simplify, Decentralize, Ensure Fairness

- Randomization can often:
  - Enable the use of a simpler algorithm
  - Cut down the amount of book-keeping
  - Support decentralized decision-making
  - Ensure fairness
- *Examples:*

  Media access protocol: Avoids need for coordination — important here, because coordination needs connectivity!

  Load balancing: Instead of maintaining centralized information about processor loads, dispatch jobs randomly.

  Congestion avoidance: Similar to load balancing

Set Theory and Probability

- A countable *sample space* $\mathcal{S}$ is a nonempty countable set.
- An *outcome* ω is an element of $\mathcal{S}$.
- A *probability function* $Pr : \mathcal{S} \longrightarrow \mathbb{R}$ is a total function such that
  - $Pr[\omega] \geq 0$ for all $\omega \in \mathcal{S}$, and
  - $\sum_{\omega \in \mathcal{S}} Pr[\omega] = 1$

Set Theory and Probability

- A countable *sample space* $\mathcal{S}$ is a nonempty countable set.
- An *outcome* ω is an element of $\mathcal{S}$.
- A *probability function* $Pr : \mathcal{S} \longrightarrow \mathbb{R}$ is a total function such that
  - $Pr[\omega] \geq 0$ for all $\omega \in \mathcal{S}$, and
  - $\sum_{\omega \in \mathcal{S}} Pr[\omega] = 1$
- An *event* E is a subset of $\mathcal{S}$. Its probability is given by:

$$Pr[E] = \sum_{\omega \in E} Pr[\omega]$$

Probability Rules from Set Theory

Many probability rules follow from the rules on set cardinality

Sum Rule: If $E_0, E_1, \ldots, E_n, \ldots$ are pairwise disjoint events, then

$Pr[\bigcup_{n \in \mathbb{N}} E_n] = \sum_{n \in \mathbb{N}} Pr[E_n]$

Probability Rules from Set Theory

Many probability rules follow from the rules on set cardinality

Sum Rule: If $E_0, E_1, \ldots, E_n, \ldots$ are pairwise disjoint events, then

$$Pr[\bigcup_{n\in\mathbb{N}} E_n] = \sum_{n\in\mathbb{N}} Pr[E_n]$$

Complement Rule: $Pr[\overline{A}] = 1 - Pr[A]$

Probability Rules from Set Theory

Many probability rules follow from the rules on set cardinality

Sum Rule: If $E_0, E_1, \ldots, E_n, \ldots$ are pairwise disjoint events, then

$$Pr[\bigcup_{n\in\mathbb{N}} E_n] = \sum_{n\in\mathbb{N}} Pr[E_n]$$

Complement Rule: $Pr[\overline{A}] = 1 - Pr[A]$

Difference Rule:

$$Pr[B - A] = Pr[B] - Pr[A \cap B]$$

Probability Rules from Set Theory

Many probability rules follow from the rules on set cardinality

Sum Rule: If $E_0, E_1, \ldots, E_n, \ldots$ are pairwise disjoint events, then

$$Pr[\bigcup_{n\in\mathbb{N}} E_n] = \sum_{n\in\mathbb{N}} Pr[E_n]$$

Complement Rule: $Pr[\overline{A}] = 1 - Pr[A]$

Difference Rule:

$$Pr[B - A] = Pr[B] - Pr[A \cap B]$$

Inclusion–Exclusion:

$$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$$

Union Bound: $Pr[A \cup B] \leq Pr[A] + Pr[B]$

Probability Rules from Set Theory

Many probability rules follow from the rules on set cardinality

Sum Rule: If $E_0, E_1, \ldots, E_n, \ldots$ are pairwise disjoint events, then

$$Pr[\bigcup_{n\in\mathbb{N}} E_n] = \sum_{n\in\mathbb{N}} Pr[E_n]$$

Complement Rule: $Pr[\overline{A}] = 1 - Pr[A]$

Difference Rule:

$$Pr[B - A] = Pr[B] - Pr[A \cap B]$$

Inclusion–Exclusion:

$$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$$

Union Bound: $Pr[A \cup B] \leq Pr[A] + Pr[B]$

Monotonicity: $A \subseteq B \to Pr[A] \leq Pr[B]$

Uniform Probability Spaces

A finite probability space $\mathcal{S}$ said to be uniform if $Pr[\omega]$ is the same for all ω. In such spaces:

$$Pr[E] = \frac{|E|}{|\mathcal{S}|}$$

We often this assumption.

Conditional Probability

- Probability of an event under a condition
- The condition limits consideration to a subset of outcomes
  - Consider this subset (rather than whole of $\mathcal{S}$) as the space of all possible outcomes

$$Pr[X|Y] = \frac{Pr[X \cap Y]}{Pr[Y]}$$

Extending Probability Rules for Conditional Probability

Product Rule 2: $Pr[E_1 \cap E_2] = Pr[E_1] \cdot Pr[E_2|E_1]$

Extending Probability Rules for Conditional Probability

Product Rule 2: $Pr[E_1 \cap E_2] = Pr[E_1] \cdot Pr[E_2 | E_1]$

Product Rule 3: $Pr[E_1 \cap E_2 \cap E_3] = Pr[E_1] \cdot Pr[E_2 | E_1] \cdot Pr[E_3 | E_1 \cap E_2]$

Extending Probability Rules for Conditional Probability

Product Rule 2: $Pr[E_1 \cap E_2] = Pr[E_1] \cdot Pr[E_2 | E_1]$

Product Rule 3: $Pr[E_1 \cap E_2 \cap E_3] = Pr[E_1] \cdot Pr[E_2 | E_1] \cdot Pr[E_3 | E_1 \cap E_2]$

Bayes' Rule: $Pr[B|A] = \dfrac{Pr[A|B] \cdot Pr[B]}{Pr[A]}$

Extending Probability Rules for Conditional Probability

Product Rule 2: $Pr[E_1 \cap E_2] = Pr[E_1] \cdot Pr[E_2|E_1]$

Product Rule 3: $Pr[E_1 \cap E_2 \cap E_3] = Pr[E_1] \cdot Pr[E_2|E_1] \cdot Pr[E_3|E_1 \cap E_2]$

Bayes' Rule: $Pr[B|A] = \dfrac{Pr[A|B] \cdot Pr[B]}{Pr[A]}$

Total Probability Law: $Pr[A] = Pr[A|E] \cdot Pr[E] + Pr[A|\overline{E}] \cdot Pr[\overline{E}]$

Total Probability Law 2: If E_i are mutually disjoint and $Pr[\bigcup E_i] = 1$ then

$$Pr[A] = \sum Pr[A|E_i] \cdot Pr[E_i]$$

Inclusion-Exclusion: $Pr[A \cup B|C] = Pr[A|C] + Pr[B|C] - Pr[A \cap B|C]$

Independence

- An event A is independent of B iff the following (equivalent) conditions hold:
  - $Pr[A|B] = Pr[A]$
  - $Pr[A \cap B] = Pr[A] \cdot Pr[B]$
  - B is independent of A
- Often, independence is an assumption.
- Definition can be generalized to 3 (or n) events. Events E_1, E_2 and E_3a are mutually independent iff all of the following hold:
  - $Pr[E_1 \cap E_2] = Pr[E_1] \cdot Pr[E_2]$
  - $Pr[E_2 \cap E_3] = Pr[E_2] \cdot Pr[E_3]$
  - $Pr[E_1 \cap E_3] = Pr[E_1] \cdot Pr[E_3]$
  - $Pr[E_1 \cap E_2 \cap E_3] = Pr[E_1] \cdot Pr[E_2] \cdot Pr[E_3]$

Coupon Collector Problem

- Suppose that your favorite cereal has a coupon inside. There are n types of coupons, but only one of them in each box. How many boxes will you have to buy before you can expect to have all of the n types?
- What is your guess?

Coupon Collector Problem

- Suppose that your favorite cereal has a coupon inside. There are n types of coupons, but only one of them in each box. How many boxes will you have to buy before you can expect to have all of the n types?
- What is your guess?
- Let us work out the expectation. Let us say that you have so far $j-1$ types of coupons, and are now looking to get to the jth type. Let X_j denote the number of boxes you need to purchase before you get the $j+1$th type.

Coupon Collector Problem

- Note $E[X_j] = 1/p_j$, where p_j is the probability of getting the *jth* coupon.

Coupon Collector Problem

- Note $E[X_j] = 1/p_j$, where p_j is the probability of getting the *jth* coupon.
- Note $p_j = (n-j)/n$, so, $E[X_j] = n/(n-j)$

Coupon Collector Problem

- Note $E[X_j] = 1/p_j$, where p_j is the probability of getting the *jth* coupon.
- Note $p_j = (n-j)/n$, so, $E[X_j] = n/(n-j)$
- We have all n types when we finish the X_{n-1} phase:

$$E[X] = \sum_{i=0}^{n-1} E[X_j] = \sum_{i=0}^{n-1} n/(n-j) = nH(n)$$

  - Note $H(n)$ is the harmonic sum, and is bounded by $\ln n$

Coupon Collector Problem

- Note $E[X_j] = 1/p_j$, where p_j is the probability of getting the *jth* coupon.
- Note $p_j = (n-j)/n$, so, $E[X_j] = n/(n-j)$
- We have all n types when we finish the X_{n-1} phase:

$$E[X] = \sum_{i=0}^{n-1} E[X_j] = \sum_{i=0}^{n-1} n/(n-j) = nH(n)$$

  - Note $H(n)$ is the harmonic sum, and is bounded by $\ln n$
- Perhaps unintuitively, you need to buy $\ln n$ cereal boxes to obtain one useful coupon.

Birthday Paradox

- What is the smallest size group where there are at least two people with the same birthday?
  - 365
  - 183
  - 61
  - 25

Birthday Problem

- The probability that two students have different birthdays: $\frac{364}{365}$
- In a class of n, there are $\binom{n}{2}$ pairs of students to consider.
  - If we assume that whether one pair shares a birthday is independent of another, we can simply multiply these probabilities

$$Pr(\text{no two persons with same birthday}) \approx \left(\frac{364}{365}\right)^{\binom{n}{2}} \approx \left(\frac{364}{365}\right)^{n^2/2}$$

- For $n = 44$, this formula yields a probability of 7%
  - $n = 23$ is enough to have better than even chance of finding two with the same birthday.

Birthday Problem: More Accurate Approach

- What is the probability of finding two people with the same birthday in this class?
- There are 365^n possible sequences of birthdays for n people
  - We assume these are all equally likely
- Number of sequences without repetition: $365 \cdot 364 \cdots (365 - (n-1))$
- Probability that no two of n persons have same birthday:

$$\frac{365}{365} \cdot \frac{365-1}{365} \cdots \frac{365-(n-1)}{365} = \left(1 - \frac{0}{365}\right)\left(1 - \frac{1}{365}\right) \cdots \left(1 - \frac{n-1}{365}\right)$$

- Use the approximation $(1-x) < e^{-x}$ to derive an upper bound:

$$Pr(\text{no two persons with same birthday}) < e^0 \cdot e^{-\frac{1}{365}} \cdot e^{-\frac{n-1}{365}} = e^{\frac{-1}{365}\sum_{i=1}^{n-1} i} = e^{\frac{-n(n-1)}{2*365}}$$

- For $n = 44$, this evaluates to 7.5%

Birthday Paradox Vs Coupon Collection

- Two sides of the same problem

  Coupon Collection: What is the minumum number of samples needed to cover every one of N values

  Birthday problem: What is the maximum number of samples that can avoid covering any value more than once?

Birthday Paradox Vs Coupon Collection

- Two sides of the same problem

  Coupon Collection: What is the minumum number of samples needed to cover every one of N values

  Birthday problem: What is the maximum number of samples that can avoid covering any value more than once?

- So, if we want enough people to ensure that every day of the year is covered as a birthday, we will need $365 \ln 365 \approx 2153$ people!
  - Almost 100 times as many as needed for one duplicate birthday!

Balls and Bins

If m balls are thrown at random into n bins:

- What should m be to have more than one ball in some bin?

Balls and Bins

If m balls are thrown at random into n bins:

- What should m be to have more than one ball in some bin?
  - Birthday problem

Balls and Bins

If m balls are thrown at random into n bins:

- What should m be to have more than one ball in some bin?
  - Birthday problem
- What should m be to have at least one ball per bin?

Balls and Bins

If m balls are thrown at random into n bins:

- What should m be to have more than one ball in some bin?
  - Birthday problem
- What should m be to have at least one ball per bin?
  - Coupon collection

Balls and Bins

If m balls are thrown at random into n bins:

- What should m be to have more than one ball in some bin?
  - Birthday problem
- What should m be to have at least one ball per bin?
  - Coupon collection
- What is the maximum number of balls in any bin?
  - Such problems arise in load-balancing, hashing, etc.

Balls and Bins: Max Occupancy

- Probability $p_{1,k}$ that the first bin receives at least k balls:
  - Choose k balls in $\binom{m}{k}$ ways
  - These k balls should fall into the first bin: prob. is $(1/n)^k$
  - Other balls may fall anywhere, i.e., probability 1:[1]

$$\binom{m}{k}\left(\frac{1}{n}\right)^k = \frac{m\cdot(m-1)\cdots(m-k+1)}{k!n^k} \leq \frac{m^k}{k!n^k}$$

[1]This is actually an upper bound, as there can be some double counting.

Balls and Bins: Max Occupancy

- Probability $p_{1,k}$ that the first bin receives at least k balls:
  - Choose k balls in $\binom{m}{k}$ ways
  - These k balls should fall into the first bin: prob. is $(1/n)^k$
  - Other balls may fall anywhere, i.e., probability 1:[1]

$$\binom{m}{k}\left(\frac{1}{n}\right)^k = \frac{m\cdot(m-1)\cdots(m-k+1)}{k!n^k} \le \frac{m^k}{k!n^k}$$

- Let $m = n$, and use Sterling's approx. $k! \approx \sqrt{2\pi k}(k/e)^k$:

$$P_k = \sum_{i=1}^{n} p_{i,k} \le n\cdot\frac{1}{k!} \le n\cdot\left(\frac{e}{k}\right)^k$$

[1] This is actually an upper bound, as there can be some double counting.

Balls and Bins: Max Occupancy

- Probability $p_{1,k}$ that the first bin receives at least k balls:
  - Choose k balls in $\binom{m}{k}$ ways
  - These k balls should fall into the first bin: prob. is $(1/n)^k$
  - Other balls may fall anywhere, i.e., probability 1:[1]

$$\binom{m}{k}\left(\frac{1}{n}\right)^k = \frac{m\cdot(m-1)\cdots(m-k+1)}{k!n^k} \le \frac{m^k}{k!n^k}$$

- Let $m = n$, and use Sterling's approx. $k! \approx \sqrt{2\pi k}(k/e)^k$:

$$P_k = \sum_{i=1}^{n} p_{i,k} \le n\cdot\frac{1}{k!} \le n\cdot\left(\frac{e}{k}\right)^k$$

- Some arithmetic simplification will show that $P_k < 1/n$ when

$$k = \frac{3\ln n}{\ln\ln n}$$

[1] This is actually an upper bound, as there can be some double counting.

Balls and Bins: Summary of Results

m balls are thrown at random into n bins:

- Min. one bin with expectation of 2 balls: $m = \sqrt{2n}$

Balls and Bins: Summary of Results

m balls are thrown at random into n bins:

- Min. one bin with expectation of 2 balls: $m = \sqrt{2n}$
- No bin expected to be empty: $m = n \ln n$

Balls and Bins: Summary of Results

m balls are thrown at random into n bins:

- Min. one bin with expectation of 2 balls: $m = \sqrt{2n}$
- No bin expected to be empty: $m = n \ln n$
- Expected number of empty bins: $ne^{-m/n}$

Balls and Bins: Summary of Results

m balls are thrown at random into n bins:

- Min. one bin with expectation of 2 balls: $m = \sqrt{2n}$
- No bin expected to be empty: $m = n \ln n$
- Expected number of empty bins: $ne^{-m/n}$
- Max. balls in any bin when $m = n$:

$$\Theta(\ln n / \ln \ln n)$$

  - This is a probabilistic bound: chance of finding any bin with higher occupancy is $1/n$ or less.
  - Note that the absolute maximum is n.

Randomized Quicksort

- Picks a pivot at random. What is its complexity?

Randomized Quicksort

- Picks a pivot at random. What is its complexity?
- If pivot index is picked uniformly at random over the interval $[l, h]$, then:
  - every array element is equally likely to be selected as the pivot
  - every partition is equally likely
  - thus, *expected* complexity of *randomized* quicksort is given by:

$$T(n) = n + \frac{1}{n}\sum_{i=1}^{n-1}(T(i) + T(n-i))$$

Randomized Quicksort

- Picks a pivot at random. What is its complexity?
- If pivot index is picked uniformly at random over the interval $[l, h]$, then:
  - every array element is equally likely to be selected as the pivot
  - every partition is equally likely
  - thus, *expected* complexity of *randomized* quicksort is given by:

$$T(n) = n + \frac{1}{n}\sum_{i=1}^{n-1}(T(i) + T(n-i))$$

Summary: Input need not be random

- Expected $O(n \log n)$ performance comes from *externally forced* randomness in picking the pivot

Cache or Page Eviction

- Caching algorithms have to evict entries when there is a miss
  - As do virtual memory systems on a page fault

Cache or Page Eviction

- Caching algorithms have to evict entries when there is a miss
  - As do virtual memory systems on a page fault
- Optimally, we should evict the "farthest in future" entry
  - But we can't predict the future!

Cache or Page Eviction

- Caching algorithms have to evict entries when there is a miss
  - As do virtual memory systems on a page fault
- Optimally, we should evict the "farthest in future" entry
  - But we can't predict the future!
- Result: many candidates for eviction. How can be avoid making bad (worst-case) choices repeatedly, even if input behaves badly?

Cache or Page Eviction

- Caching algorithms have to evict entries when there is a miss
  - As do virtual memory systems on a page fault
- Optimally, we should evict the “farthest in future” entry
  - But we can’t predict the future!
- Result: many candidates for eviction. How can be avoid making bad (worst-case) choices repeatedly, even if input behaves badly?
- Approach: pick one of the candidates at random!

Hash Tables

- A data structure for implementing:

  Dictionaries: Fast look up of a record based on a key.

  Sets: Fast membership check.

Hash Tables

- A data structure for implementing:

  Dictionaries: Fast look up of a record based on a key.

  Sets: Fast membership check.

- Support expected $O(1)$ time *lookup*, *insert*, and *delete*

Hash Tables

- A data structure for implementing:

  Dictionaries: Fast look up of a record based on a key.

  Sets: Fast membership check.

- Support expected $O(1)$ time *lookup*, *insert*, and *delete*

- Hash table entries may be:

  fat: store a pair $(key, object)$

  lean: store pointer to object containing key

Hash Tables

- A data structure for implementing:

  Dictionaries: Fast look up of a record based on a key.

  Sets: Fast membership check.

- Support expected $O(1)$ time *lookup*, *insert*, and *delete*

- Hash table entries may be:

  fat: store a pair $(key, object)$

  lean: store pointer to object containing key

- Two main questions:
  - *How to avoid $O(n)$ worst case behavior?*
  - How to ensure *average case performance* can be realized *for arbitrary distribution of keys?*

Hash Table Implementation

Direct access: A fancy name for arrays. Not applicable in most cases where the universe $\mathcal{U}$ of keys is very large.

Hash Table Implementation

Direct access: A fancy name for arrays. Not applicable in most cases where the universe $\mathcal{U}$ of keys is very large.

Index based on hash: Given a hash function h (fixed for the entire table) and a key x, use $h(x)$ to index into an array A.

Hash Table Implementation

Direct access: A fancy name for arrays. Not applicable in most cases where the universe $\mathcal{U}$ of keys is very large.

Index based on hash: Given a hash function h (fixed for the entire table) and a key x, use $h(x)$ to index into an array A.

- Use $A[h(x) \mod s]$, where s is the size of array
  - Sometimes, we fold the mod operation into h.

Hash Table Implementation

Direct access: A fancy name for arrays. Not applicable in most cases where the universe $\mathcal{U}$ of keys is very large.

Index based on hash: Given a hash function h (fixed for the entire table) and a key x, use $h(x)$ to index into an array A.

- Use $A[h(x) \mod s]$, where s is the size of array
  - Sometimes, we fold the mod operation into h.
- Array elements typically called *buckets*

Hash Table Implementation

Direct access: A fancy name for arrays. Not applicable in most cases where the universe $\mathcal{U}$ of keys is very large.

Index based on hash: Given a hash function h (fixed for the entire table) and a key x, use $h(x)$ to index into an array A.

- Use $A[h(x) \mod s]$, where s is the size of array
  - Sometimes, we fold the mod operation into h.
- Array elements typically called *buckets*
- *Collisions bound to occur* since $s \ll |\mathcal{U}|$
  - Either $h(x) = h(y)$, or
  - $h(x) \neq h(y)$ but $h(x) \equiv h(y) \pmod{s}$

Collisions in Hash tables

- *Load factor α:* Ratio of number of keys to number of buckets

Collisions in Hash tables

- *Load factor α:* Ratio of number of keys to number of buckets
- *If* keys were random:
  - What is the max α if we want ≤ 1 collisions in the table?
  - If $\alpha = 1$, what is the maximum number of collisions to expect?

Collisions in Hash tables

- *Load factor α:* Ratio of number of keys to number of buckets
- *If* keys were random:
  - What is the max α if we want ≤ 1 collisions in the table?
  - If $\alpha = 1$, what is the maximum number of collisions to expect?
- Both questions can be answered from balls-and-bins results: $1/\sqrt{n}$, and $O(\ln n/\ln\ln n)$
- **Real world keys are not random.** Your hash table implementation needs to achieve its performance goals independent of this distribution.

Chained Hash Table

- Each bucket is a linked list.
- Any key that hashes to a bucket is inserted into that bucket.
- What is the *average* search time, as a function of α?

Chained Hash Table

- Each bucket is a linked list.
- Any key that hashes to a bucket is inserted into that bucket.
- What is the *average* search time, as a function of α?
  - It is $1 + \alpha$ if:
    - you assume that the distribution of lookups is independent of the table entries, OR,
    - the chains are not too long (i.e., α is small)

Open addressing

- If there is a collision, probe other empty slots

  Linear probing: If $h(x)$ is occupied, try $h(x) + i$ for $i = 1, 2, \ldots$

  Binary probing: Try $h(x) \oplus i$, where $\oplus$ stands for exor.

  Quadratic probing: For ith probe, use $h(x) + c_1 i + c_2 i^2$

Open addressing

- If there is a collision, probe other empty slots

  Linear probing: If $h(x)$ is occupied, try $h(x) + i$ for $i = 1, 2, \ldots$

  Binary probing: Try $h(x) \oplus i$, where $\oplus$ stands for exor.

  Quadratic probing: For ith probe, use $h(x) + c_1 i + c_2 i^2$

- Criteria for secondary probes

  Completeness: Should cycle through all possible slots in table

  Clustering: Probe sequences shouldn't coalesce to long chains

  Locality: Preserve locality; typically conflicts with clustering.

Open addressing

- If there is a collision, probe other empty slots

  Linear probing: If $h(x)$ is occupied, try $h(x) + i$ for $i = 1, 2, \ldots$

  Binary probing: Try $h(x) \oplus i$, where $\oplus$ stands for exor.

  Quadratic probing: For ith probe, use $h(x) + c_1 i + c_2 i^2$

- Criteria for secondary probes

  Completeness: Should cycle through all possible slots in table

  Clustering: Probe sequences shouldn't coalesce to long chains

  Locality: Preserve locality; typically conflicts with clustering.

- Average search time can be $O(1/(1-\alpha)^2)$ for linear probing, and $O(1/(1-\alpha))$ for quadratic probing.

Chaining Vs Open Addressing

- Chaining leads to fewer collisions
  - Clustering causes more collisions w/ open addressing for same α
  - However, for lean tables, open addressing uses half the space of chaining, so you can use a much lower α for same space usage.

Chaining Vs Open Addressing

- Chaining leads to fewer collisions
  - Clustering causes more collisions w/ open addressing for same α
  - However, for lean tables, open addressing uses half the space of chaining, so you can use a much lower α for same space usage.
- Chaining is more tolerant of "lumpy" hash functions
  - For instance, if $h(x)$ and $h(x+1)$ are often very close, open hashing can experience longer chains when inputs are closely spaced.
  - Hash functions for open-hashing having to be selected very carefully

Chaining Vs Open Addressing

- Chaining leads to fewer collisions
  - Clustering causes more collisions w/ open addressing for same α
  - However, for lean tables, open addressing uses half the space of chaining, so you can use a much lower α for same space usage.
- Chaining is more tolerant of "lumpy" hash functions
  - For instance, if $h(x)$ and $h(x+1)$ are often very close, open hashing can experience longer chains when inputs are closely spaced.
  - Hash functions for open-hashing having to be selected very carefully
- Linked lists are not cache-friendly
  - Can be mitigated w/ arrays for buckets instead of linked lists

Chaining Vs Open Addressing

- Chaining leads to fewer collisions
  - Clustering causes more collisions w/ open addressing for same α
  - However, for lean tables, open addressing uses half the space of chaining, so you can use a much lower α for same space usage.
- Chaining is more tolerant of "lumpy" hash functions
  - For instance, if $h(x)$ and $h(x+1)$ are often very close, open hashing can experience longer chains when inputs are closely spaced.
  - Hash functions for open-hashing having to be selected very carefully
- Linked lists are not cache-friendly
  - Can be mitigated w/ arrays for buckets instead of linked lists
- Not all quadratic probes cover all slots (but some can)

Resizing

- Hard to predict the right size for hash table in advance
  - Ideally, $0.5 \leq \alpha \leq 1$, so we need an accurate estimate

Resizing

- Hard to predict the right size for hash table in advance
  - Ideally, $0.5 \leq \alpha \leq 1$, so we need an accurate estimate
- *It is stupid to ask programmers to guess the size*
  - Without a good basis, only terrible guesses are possible

Resizing

- Hard to predict the right size for hash table in advance
  - Ideally, $0.5 \leq \alpha \leq 1$, so we need an accurate estimate
- *It is stupid to ask programmers to guess the size*
  - Without a good basis, only terrible guesses are possible
- *Right solution:* Resize tables automatically.
  - When α becomes too large (or small), rehash into a bigger (or smaller) table

Resizing

- Hard to predict the right size for hash table in advance
  - Ideally, $0.5 \leq \alpha \leq 1$, so we need an accurate estimate
- *It is stupid to ask programmers to guess the size*
  - Without a good basis, only terrible guesses are possible
- *Right solution:* Resize tables automatically.
  - When α becomes too large (or small), rehash into a bigger (or smaller) table
  - Rehashing is $O(n)$, but if you increase size by a factor, then amortized cost is still $O(1)$

Resizing

- Hard to predict the right size for hash table in advance
  - Ideally, $0.5 \leq \alpha \leq 1$, so we need an accurate estimate
- *It is stupid to ask programmers to guess the size*
  - Without a good basis, only terrible guesses are possible
- *Right solution:* Resize tables automatically.
  - When α becomes too large (or small), rehash into a bigger (or smaller) table
  - Rehashing is $O(n)$, but if you increase size by a factor, then amortized cost is still $O(1)$
  - Exercise: How to ensure amortized $O(1)$ cost when you resize up as well as down?

Average Vs Worst Case

- Worst case search time is $O(n)$ for a table of size n

Average Vs Worst Case

- Worst case search time is $O(n)$ for a table of size n
- *With hash tables, it is all about avoiding the worst case, and achieving the average case*

Average Vs Worst Case

- Worst case search time is $O(n)$ for a table of size n
- *With hash tables, it is all about avoiding the worst case, and achieving the average case*
- Two main chalenges:
  - *Input is not random,* e.g., names or IP addresses.
  - Even when input is random, h may cause "lumping," or non-uniform dispersal of $\mathcal{U}$ to the set $\{1, \ldots, n\}$

Average Vs Worst Case

- Worst case search time is $O(n)$ for a table of size n
- *With hash tables, it is all about avoiding the worst case, and achieving the average case*
- Two main chalenges:
  - *Input is not random,* e.g., names or IP addresses.
  - Even when input is random, h may cause “lumping,” or non-uniform dispersal of $\mathcal{U}$ to the set $\{1, \ldots, n\}$
- Two main techniques

  Universal hashing

  Perfect hashing

Universal Hashing

- No single hash function can be good on all inputs
  - Any function $\mathcal{U} \to \{1, \ldots, n\}$ must map $|\mathcal{U}|/n$ inputs to same value!

Note: $|\mathcal{U}|$ can be much, much larger than n.

Universal Hashing

- No single hash function can be good on all inputs
  - Any function $\mathcal{U} \to \{1, \ldots, n\}$ must map $|\mathcal{U}|/n$ inputs to same value!

Note: $|\mathcal{U}|$ can be much, much larger than n.

Definition

A family of hash functions $\mathcal{H}$ is universal if

$$Pr_{h \in \mathcal{H}}[h(x) = h(y)] = \frac{1}{n} \quad \textit{for all } x \neq y$$

Universal Hashing

- No single hash function can be good on all inputs
  - Any function $\mathcal{U} \to \{1, \ldots, n\}$ must map $|\mathcal{U}|/n$ inputs to same value!

Note: $|\mathcal{U}|$ can be much, much larger than n.

Definition

A family of hash functions $\mathcal{H}$ is universal if

$$Pr_{h \in \mathcal{H}}[h(x) = h(y)] = \frac{1}{n} \quad \textit{for all } x \neq y$$

Meaning: If we pick h at random from the family $\mathcal{H}$, then, probability of collisions is the same for any two elements.

Contrast with non-universal hash functions such as

$$h(x) = ax \text{ mod } n, \qquad (a \text{ is chosen at random})$$

Note y and $y + kn$ collide with a probability of 1 *for every* a.

Universal Hashing Using Multiplication

Observation (Multiplication Modulo Prime)

If p is a prime and $0 < a < p$

- $\{1a, 2a, 3a, \ldots, (p-1)a\} = \{1, 2, \ldots, p-1\}$ *(mod p)*
- $\forall a \, \exists b \; ab \equiv 1 \pmod{p}$

Universal Hashing Using Multiplication

Observation (Multiplication Modulo Prime)

If p is a prime and $0 < a < p$

- $\{1a, 2a, 3a, \ldots, (p-1)a\} = \{1, 2, \ldots, p-1\}$ *(mod p)*
- $\forall a \, \exists b \; ab \equiv 1 \pmod{p}$

Prime multiplicative hashing

Let the key $x \in \mathcal{U}$, $p > |\mathcal{U}|$ be prime, and $0 < r < p$ be random. Then

$$h(x) = (rx \mod p) \mod n$$

is universal.

Prove: $Pr[h(x) = h(y)] = \frac{1}{n}$, for $x \neq y$

Universality of prime multiplicative hashing

- Need to show $Pr[h(x) = h(y)] = \frac{1}{n}, \quad \text{for } x \neq y$
- $h(x) = h(y)$ means $(rx \textbf{ mod } p) \textbf{ mod } n = (ry \textbf{ mod } p) \textbf{ mod } n$
- Note $a \textbf{ mod } n = b \textbf{ mod } n$ means $a = b + kn$ for some integer k. Using this, we eliminate $\textbf{mod } n$ from above equation to get:

$$\begin{aligned} rx \textbf{ mod } p &= kn + ry \textbf{ mod } p, \text{ where } k \leq \lfloor p/n \rfloor \\ rx &\equiv kn + ry \pmod{p} \\ r(x-y) &\equiv kn \pmod{p} \\ r &\equiv kn(x-y)^{-1} \pmod{p} \end{aligned}$$

- So, x, y collide if $r = n(x-y)^{-1}, 2n(x-y)^{-1}, \ldots, \lfloor p/n \rfloor n(x-y)^{-1}$
- In other words, x and y collide for p/n out of p possible values of r, i.e., collision probability is $1/n$

Binary multiplicative hashing

- Faster: avoids need for computing modulo prime

Binary multiplicative hashing

- Faster: avoids need for computing modulo prime
- When $|\mathcal{U}| < 2^w$, $n = 2^l$ and a an odd random number

$$h(x) = \left\lfloor \frac{ax \mod 2^w}{2^{w-l}} \right\rfloor$$

Binary multiplicative hashing

- Faster: avoids need for computing modulo prime
- When $|\mathcal{U}| < 2^w$, $n = 2^l$ and a an odd random number

$$h(x) = \left\lfloor \frac{ax \mod 2^w}{2^{w-l}} \right\rfloor$$

- Can be implemented efficiently if w is the wordsize:

```
(a*x) >> (WORDSIZE-HASHBITS)
```

Binary multiplicative hashing

- Faster: avoids need for computing modulo prime
- When $|\mathcal{U}| < 2^w$, $n = 2^l$ and a an odd random number

$$h(x) = \left\lfloor \frac{ax \mod 2^w}{2^{w-l}} \right\rfloor$$

- Can be implemented efficiently if w is the wordsize:

```
(a*x) >> (WORDSIZE-HASHBITS)
```

- Scheme is near-universal: collision probability is $O(1)/2^l$

Prime Multiplicative Hash for Vectors

Let p be a prime number, and the key x be a vector $[x_1, \ldots, x_k]$ where $0 \leq x_i < p$. Let

$$h(x) = \sum_{i=1}^{k} r_i x_i \pmod{p}$$

If $0 < r_i < p$ are chosen at random, then h is universal.

Prime Multiplicative Hash for Vectors

Let p be a prime number, and the key x be a vector $[x_1, \ldots, x_k]$ where $0 \leq x_i < p$. Let

$$h(x) = \sum_{i=1}^{k} r_i x_i \pmod{p}$$

If $0 < r_i < p$ are chosen at random, then h is universal.

- Strings can also be handled like vectors, or alternatively, as a polynomial evaluated at a random point a, with p a prime:

$$h(x) = \sum_{i=0}^{l} x_i a^i \ \ mod \ p$$

Universality of multiplicative hashing for vectors

- Since $x \neq y$, there exists an i such that $x_i \neq y_i$

Universality of multiplicative hashing for vectors

- Since $x \neq y$, there exists an i such that $x_i \neq y_i$
- When collision occurs, $\sum_{j=1}^{k} r_j x_j = \sum_{j=1}^{k} r_j y_j \pmod p$

Universality of multiplicative hashing for vectors

- Since $x \neq y$, there exists an i such that $x_i \neq y_i$
- When collision occurs, $\sum_{j=1}^{k} r_j x_j = \sum_{j=1}^{k} r_j y_j \pmod p$
- Rearranging, $\sum_{j \neq i} r_j(x_j - y_j) = r_i(y_i - x_i) \pmod p$

Universality of multiplicative hashing for vectors

- Since $x \neq y$, there exists an i such that $x_i \neq y_i$
- When collision occurs, $\sum_{j=1}^{k} r_j x_j = \sum_{j=1}^{k} r_j y_j \pmod p$
- Rearranging, $\sum_{j \neq i} r_j (x_j - y_j) = r_i (y_i - x_i) \pmod p$
- The lhs evaluates to some c, and we need to estimate the probability that rhs evaluates to this c

Universality of multiplicative hashing for vectors

- Since $x \neq y$, there exists an i such that $x_i \neq y_i$
- When collision occurs, $\sum_{j=1}^{k} r_j x_j = \sum_{j=1}^{k} r_j y_j \pmod p$
- Rearranging, $\sum_{j \neq i} r_j(x_j - y_j) = r_i(y_i - x_i) \pmod p$
- The lhs evaluates to some c, and we need to estimate the probability that rhs evaluates to this c
- Using multiplicative inverse property, we see that $r_i = c(y_i - x_i)^{-1} \pmod p$.

Universality of multiplicative hashing for vectors

- Since $x \neq y$, there exists an i such that $x_i \neq y_i$
- When collision occurs, $\sum_{j=1}^{k} r_j x_j = \sum_{j=1}^{k} r_j y_j \pmod{p}$
- Rearranging, $\sum_{j \neq i} r_j(x_j - y_j) = r_i(y_i - x_i) \pmod{p}$
- The lhs evaluates to some c, and we need to estimate the probability that rhs evaluates to this c
- Using multiplicative inverse property, we see that $r_i = c(y_i - x_i)^{-1} \pmod{p}$.
- Since $y_i, x_i < p$, it is easy to see from this equation that the collision-causing value of r_i is distinct for distinct y_i.

Universality of multiplicative hashing for vectors

- Since $x \neq y$, there exists an i such that $x_i \neq y_i$
- When collision occurs, $\sum_{j=1}^{k} r_j x_j = \sum_{j=1}^{k} r_j y_j \pmod p$
- Rearranging, $\sum_{j \neq i} r_j(x_j - y_j) = r_i(y_i - x_i) \pmod p$
- The lhs evaluates to some c, and we need to estimate the probability that rhs evaluates to this c
- Using multiplicative inverse property, we see that $r_i = c(y_i - x_i)^{-1} \pmod p$.
- Since $y_i, x_i < p$, it is easy to see from this equation that the collision-causing value of r_i is distinct for distinct y_i.
- Viewed another way, exactly one of p choices of r_i would cause a collision between x_i and y_i, i.e., $Pr_h[h(x) = h(y)] = 1/p$

Perfect hashing

Static: Pick a hash function (or set of functions) that avoids collisions for a given set of keys

Perfect hashing

Static: Pick a hash function (or set of functions) that avoids collisions for a given set of keys

Dynamic: Keys need not be static.

Approach 1: Use $O(n^2)$ storage. Expected collision on n items is 0. But too wasteful of storage.

Don't forget: more memory usually means less performance due to cache effects.

Perfect hashing

Static: Pick a hash function (or set of functions) that avoids collisions for a given set of keys

Dynamic: Keys need not be static.

Approach 1: Use $O(n^2)$ storage. Expected collision on n items is 0. But too wasteful of storage.

Don't forget: more memory usually means less performance due to cache effects.

Approach 2: Use a secondary hash table for each bucket of size n_i^2, where n_i is the number of elements in the bucket.

Uses only $O(n)$ storage, *if h is universal*

Hashing Summary

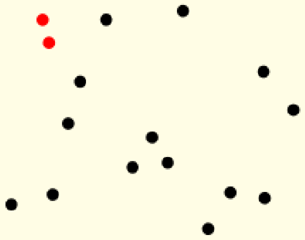
- Excellent average case performance
  - Pointer chasing is expensive on modern hardware, so improvement from $O(\log n)$ of binary trees to expected $O(1)$ for hash tables is significant.

Hashing Summary

- Excellent average case performance
  - Pointer chasing is expensive on modern hardware, so improvement from $O(\log n)$ of binary trees to expected $O(1)$ for hash tables is significant.
- But all benefits will be reversed if collisions occur too often
  - Universal hashing is a way to ensure expected average case *even when input is not random.*
- Perfect hashing can provide efficient performance even in the worst case, but the benefits are likely small in practice.

Finding closest pair of points

Problem: Given a set of n points in a d-dimensional space, identify the two that have the smallest Euclidean distance between them.



Applications: A central problem in graphics, vision, air-traffic control, navigation, molecular modeling, and so on.

Images from Wikipedia Commons

Randomized Closest Pair: Key Ideas

- Divide the plane into small squares, hash points into them
  - Pairwise comparisons can be limited to points within the squares very closeby

Randomized Closest Pair: Key Ideas

- Divide the plane into small squares, hash points into them
  - Pairwise comparisons can be limited to points within the squares very closeby
- Process the points in some random order
  - Maintain min. distance δ among points processed so far.
  - Update δ as more points are processed

Randomized Closest Pair: Key Ideas

- Divide the plane into small squares, hash points into them
  - Pairwise comparisons can be limited to points within the squares very closeby
- Process the points in some random order
  - Maintain min. distance δ among points processed so far.
  - Update δ as more points are processed
- At any point, the "small squares" have a size of $\delta/2$
  - At most one point per square (or else points are closer than δ)
  - Points closer than δ will at most be two squares from each other
    - Only constant number of points to consider
  - Requires rehashing all processed points when δ is updated.

Randomized Closest Pair: Analysis

- Correctness is relatively clear, so we focus on performance
- Two main concerns

Randomized Closest Pair: Analysis

- Correctness is relatively clear, so we focus on performance
- Two main concerns

  Storage: # of squares is $1/\delta^2$, which can be very large

Randomized Closest Pair: Analysis

- Correctness is relatively clear, so we focus on performance
- Two main concerns

  Storage: # of squares is $1/\delta^2$, which can be very large

  - Use a dictionary (hash table) that stores up to n points, and maps $(2x_i/\delta, 2y_i/\delta)$ to $\{1, \ldots, n\}$

Randomized Closest Pair: Analysis

- Correctness is relatively clear, so we focus on performance
- Two main concerns

Storage: # of squares is $1/\delta^2$, which can be very large

  - Use a dictionary (hash table) that stores up to n points, and maps $(2x_i/\delta, 2y_i/\delta)$ to $\{1, ..., n\}$
  - To process a point (x_j, y_j)
    - look up the dictionary at $(x_j/\delta \pm 2, y_j/\delta \pm 2)$
    - insert if it is not closer than δ

Randomized Closest Pair: Analysis

- Correctness is relatively clear, so we focus on performance
- Two main concerns

  Storage: # of squares is $1/\delta^2$, which can be very large
  - Use a dictionary (hash table) that stores up to n points, and maps $(2x_i/\delta, 2y_i/\delta)$ to $\{1, \ldots, n\}$
  - To process a point (x_j, y_j)
    - look up the dictionary at $(x_j/\delta \pm 2, y_j/\delta \pm 2)$
    - insert if it is not closer than δ

  Rehashing points: If closer than δ — very expensive.

Randomized Closest Pair: Analysis

- Correctness is relatively clear, so we focus on performance
- Two main concerns

  Storage: # of squares is $1/\delta^2$, which can be very large

  - Use a dictionary (hash table) that stores up to n points, and maps $(2x_i/\delta, 2y_i/\delta)$ to $\{1, \ldots, n\}$
  - To process a point (x_j, y_j)
    - look up the dictionary at $(x_j/\delta \pm 2, y_j/\delta \pm 2)$
    - insert if it is not closer than δ

  Rehashing points: If closer than δ — very expensive.

- Total runtime can all be "charged" to insert operations,
  - incl. those performed during rehashing

  so we will focus on estimating inserts.

Randomized Closest Pair: # of Inserts

Theorem

If random variable X_i denotes the likelihood of needing to rehash after processing k points, then

$$X_i \leq \frac{2}{i}$$

Randomized Closest Pair: # of Inserts

Theorem

If random variable X_i denotes the likelihood of needing to rehash after processing k points, then

$$X_i \leq \frac{2}{i}$$

- Let $p_1, p_2, \ldots, p_i$ be the points processed so far, and p and q be the closest among these

Randomized Closest Pair: # of Inserts

Theorem

If random variable X_i denotes the likelihood of needing to rehash after processing k points, then

$$X_i \leq \frac{2}{i}$$

- Let $p_1, p_2, \ldots, p_i$ be the points processed so far, and p and q be the closest among these
- Rehashing is needed while processing p_i if $p_i = p$ or $p_i = q$

Randomized Closest Pair: # of Inserts

Theorem

If random variable X_i denotes the likelihood of needing to rehash after processing k points, then

$$X_i \leq \frac{2}{i}$$

- Let $p_1, p_2, \ldots, p_i$ be the points processed so far, and p and q be the closest among these
- Rehashing is needed while processing p_i if $p_i = p$ or $p_i = q$
- Since points are processed in random order, there is a $2/i$ probability that p_i is one of p or q

Randomized Closest Pair: # of Inserts

Theorem

The expected number of inserts is $3n$.

Randomized Closest Pair: # of Inserts

Theorem

The expected number of inserts is $3n$.

- Processing of p_i involves
  - i inserts if rehashing takes place, and 1 insert otherwise

Randomized Closest Pair: # of Inserts

Theorem

The expected number of inserts is $3n$.

- Processing of p_i involves
  - i inserts if rehashing takes place, and 1 insert otherwise
- So, expected inserts for processing p_i is

$$i \cdot X_i + 1 \cdot (1 - X_i) = 1 + (i - 1) \cdot X_i = 1 + \frac{2(i-1)}{i} \leq 3$$

Randomized Closest Pair: # of Inserts

Theorem

The expected number of inserts is $3n$.

- Processing of p_i involves
  - i inserts if rehashing takes place, and 1 insert otherwise
- So, expected inserts for processing p_i is

$$i \cdot X_i + 1 \cdot (1 - X_i) = 1 + (i-1) \cdot X_i = 1 + \frac{2(i-1)}{i} \leq 3$$

- Upper bound on expected inserts is thus $3n$

Randomized Closest Pair: # of Inserts

Theorem

The expected number of inserts is $3n$.

- Processing of p_i involves
  - i inserts if rehashing takes place, and 1 insert otherwise
- So, expected inserts for processing p_i is

$$i \cdot X_i + 1 \cdot (1 - X_i) = 1 + (i - 1) \cdot X_i = 1 + \frac{2(i-1)}{i} \leq 3$$

- Upper bound on expected inserts is thus $3n$

Look Ma! I have a linear-time randomized closest pair algorithm—And it is not even probabilistic!

Probabilistic Algorithms

- Algorithms that produce the correct answer with some probability
- By re-running the algorithm many times, we can increase the probability to be arbitrarily close to 1.0.

Bloom Filters

- To resolve collisions, hash tables have to store keys: $O(mw)$ bits, where w is the number of bits in the key

Bloom Filters

- To resolve collisions, hash tables have to store keys: $O(mw)$ bits, where w is the number of bits in the key
- What if you want to store very large keys?

Bloom Filters

- To resolve collisions, hash tables have to store keys: $O(mw)$ bits, where w is the number of bits in the key
- What if you want to store very large keys?
- *Radical idea:* Don't store the key in the table!
  - Potentially w-fold space reduction

Bloom Filters

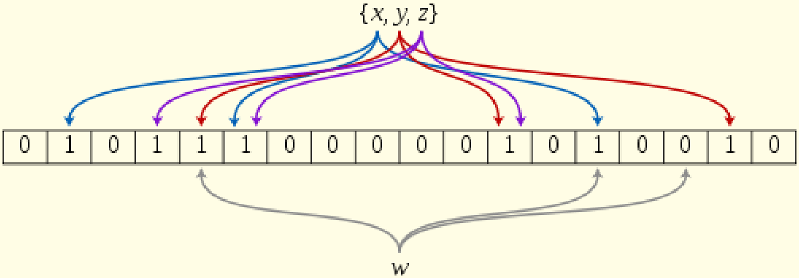
- To reduce collisions, use multiple hash functions $h_1, \ldots, h_k$

Bloom Filters

- To reduce collisions, use multiple hash functions $h_1, \ldots, h_k$
- Hash table is simply a bitvector $B[1..m]$

Bloom Filters

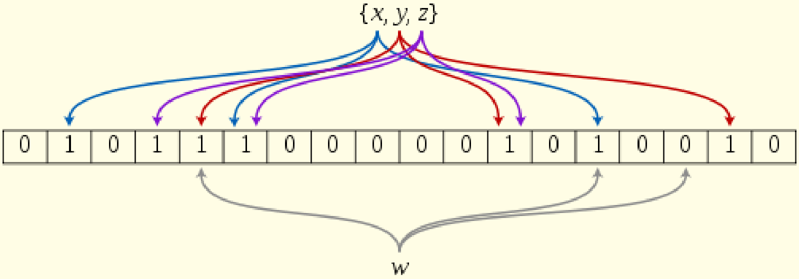
- To reduce collisions, use multiple hash functions $h_1, \ldots, h_k$
- Hash table is simply a bitvector $B[1..m]$
- To insert key x, set $B[h_1(x)], B[h_2(x)], \ldots, B[h_k(x)]$



Images from Wikipedia Commons

Bloom Filters

- To reduce collisions, use multiple hash functions $h_1, \ldots, h_k$
- Hash table is simply a bitvector $B[1..m]$
- To insert key x, set $B[h_1(x)], B[h_2(x)], \ldots, B[h_k(x)]$

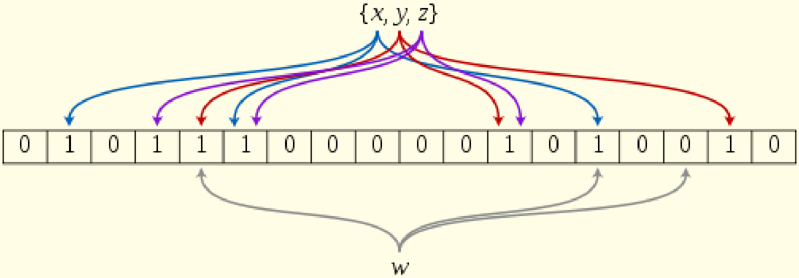


Images from Wikipedia Commons

- Membership check for y: all $B[h_i(y)]$ should be set
  - No false negatives, but false positives possible

Bloom Filters

- To reduce collisions, use multiple hash functions $h_1, \ldots, h_k$
- Hash table is simply a bitvector $B[1..m]$
- To insert key x, set $B[h_1(x)], B[h_2(x)], \ldots, B[h_k(x)]$



Images from Wikipedia Commons

- Membership check for y: all $B[h_i(y)]$ should be set
  - No false negatives, but false positives possible
- No deletions possible in the current algorithm.

Bloom Filters: False positives

- Prob. that a bit is *not* set by h_1 on inserting a key is $(1 - 1/m)$

Bloom Filters: False positives

- Prob. that a bit is *not* set by h_1 on inserting a key is $(1 - 1/m)$
    - The probability it is not set by any h_i is $(1 - 1/m)^k$

Bloom Filters: False positives

- Prob. that a bit is *not* set by h_1 on inserting a key is $(1 - 1/m)$
  - The probability it is not set by any h_i is $(1 - 1/m)^k$
  - The probability it is not set after r key inserts is $(1 - 1/m)^{kr} \approx e^{-kr/m}$

Bloom Filters: False positives

- Prob. that a bit is *not* set by h_1 on inserting a key is $(1 - 1/m)$
  - The probability it is not set by any h_i is $(1 - 1/m)^k$
  - The probability it is not set after r key inserts is $(1 - 1/m)^{kr} \approx e^{-kr/m}$
- Complementing, the prob. p that a certain bit is set is $1 - e^{-kr/m}$

Bloom Filters: False positives

- Prob. that a bit is *not* set by h_1 on inserting a key is $(1 - 1/m)$
  - The probability it is not set by any h_i is $(1 - 1/m)^k$
  - The probability it is not set after r key inserts is $(1 - 1/m)^{kr} \approx e^{-kr/m}$
- Complementing, the prob. p that a certain bit is set is $1 - e^{-kr/m}$
- For a false positive on a key y, all the bits that it hashes to should be a 1. This happens with probability

$$\left(1 - e^{-kr/m}\right)^k = (1 - p)^k$$

Bloom Filters

- Note: $n = m/r$ is the storage (in bits) used per key.
- So, we can rewrite the FP equation as:

$$\left(1 - e^{-kr/m}\right)^k = \left(1 - e^{-k/n}\right)^k$$

Bloom Filters

- Note: $n = m/r$ is the storage (in bits) used per key.
- So, we can rewrite the FP equation as:

$$\left(1 - e^{-kr/m}\right)^k = \left(1 - e^{-k/n}\right)^k$$

- Optimal value of k can be shown to be $n \ln 2$.
  - The FP rate simplifies to $0.5^{n \ln 2} = 0.619^n$
- A Bloom filter that uses just 8 bits per key to store an *arbitrary sized key* will have an FP rate of 2%

Bloom Filters

- Note: $n = m/r$ is the storage (in bits) used per key.
- So, we can rewrite the FP equation as:

$$\left(1 - e^{-kr/m}\right)^k = \left(1 - e^{-k/n}\right)^k$$

- Optimal value of k can be shown to be $n \ln 2$.
  - The FP rate simplifies to $0.5^{n \ln 2} = 0.619^n$
- A Bloom filter that uses just 8 bits per key to store an *arbitrary sized key* will have an FP rate of 2%
- *Important:* Bloom filters can be used as a prefilter, e.g., if actual keys are in secondary storage (e.g., files or internet repositories)

Using arithmetic for substring matching

Problem: Given strings $T[1..n]$ and $P[1..m]$, find occurrences of P in T in $O(n+m)$ time.

Idea: To simplify presentation, assume P, T range over `[0-9]`

- Interpret $P[1..m]$ as digits of a number

$$p = 10^{m-1}P[1] + 10^{m-2}P[2] + \cdots 10^{m-m}P[m]$$

- Similarly, interpret $T[i..(i+m-1)]$ as the number t_i
- Note: P is a substring of T at i iff $p = t_i$
- To get t_{i+1}, shift $T[i]$ out of t_i, and shift in $T[i+m]$:

$$t_{i+1} = (t_i - 10^{m-1}T[i]) \cdot 10 + T[i+m]$$

We have an $O(n+m)$ algorithm. Almost: we still need to figure out how to operate on m-digit numbers in constant time!

Rabin-Karp Fingerprinting

Key Idea

- Instead of working with m-digit numbers,
- perform all arithmetic modulo a *random* prime number q,
- where $q > m^2$ fits within wordsize

- All observations made on previous slide still hold
  - Except that $p = t_i$ does not guarantee a match
  - Typically, we expect matches to be infrequent, so we can use $O(m)$ exact-matching algorithm to confirm probable matches.

Carter-Wegman-Rabin-Karp Algorithm

Difficulty with Rabin-Karp: Need to generate random primes, which is not an efficient task.

New Idea: Make the radix random, as opposed to the modulus

- We still compute modulo a prime q, but it is not random.

Alternative interpretation: We treat P as a polynomial

$$p(x) = \sum_{i=1}^{m} P[m-i] \cdot x^i$$

and evaluate this polynomial at a randomly chosen value of x

Like any probabilistic algorithm we can increase correctness probability by repeating the algorithm with different randoms.

- Different prime numbers for Rabin-Karp
- Different values of x for CWRK

Carter-Wegman-Rabin-Karp Algorithm

$$p(x) = \sum_{i=1}^{m} P[m-i] \cdot x^i$$

Random choice does not imply high probability of being right.

- You need to explicitly establish correctness probability.

Carter-Wegman-Rabin-Karp Algorithm

$$p(x) = \sum_{i=1}^{m} P[m-i] \cdot x^i$$

Random choice does not imply high probability of being right.

- You need to explicitly establish correctness probability.

So, what is the likelihood of false matches?

- A false match occurs if $p_1(x) = p_2(x)$, i.e., $p_1(x) - p_2(x) = p_3(x) = 0$.

Carter-Wegman-Rabin-Karp Algorithm

$$p(x) = \sum_{i=1}^{m} P[m-i] \cdot x^i$$

Random choice does not imply high probability of being right.

- You need to explicitly establish correctness probability.

So, what is the likelihood of false matches?

- A false match occurs if $p_1(x) = p_2(x)$, i.e., $p_1(x) - p_2(x) = p_3(x) = 0$.
- Arithmetic modulo prime defines a *field*, so an mth degree polynomial has $m + 1$ roots.

Carter-Wegman-Rabin-Karp Algorithm

$$p(x) = \sum_{i=1}^{m} P[m-i] \cdot x^i$$

Random choice does not imply high probability of being right.

- You need to explicitly establish correctness probability.

So, what is the likelihood of false matches?

- A false match occurs if $p_1(x) = p_2(x)$, i.e., $p_1(x) - p_2(x) = p_3(x) = 0$.
- Arithmetic modulo prime defines a *field*, so an mth degree polynomial has $m+1$ roots.
- Thus, $(m+1)/q$ of the q (recall q is the prime number used for performing modulo arithmetic) possible choices of x will result in a false match, i.e., probability of false positive = $(m+1)/q$

Primality Testing

Fermat's Theorem

$$a^{p-1} \equiv 1 \pmod{p}$$

Primality Testing

Fermat's Theorem

$a^{p-1} \equiv 1 \pmod{p}$

- Recall $\{1a, 2a, 3a, \ldots, (p-1)a\} \equiv \{1, 2, \ldots, p-1\} \pmod{p}$

Primality Testing

Fermat's Theorem

$a^{p-1} \equiv 1 \pmod{p}$

- Recall $\{1a, 2a, 3a, \ldots, (p-1)a\} \equiv \{1, 2, \ldots, p-1\} \pmod{p}$
- Multiply all elements of both sides:

$$(p-1)!a^{p-1} \equiv (p-1)! \pmod{p}$$

Primality Testing

Fermat's Theorem

$a^{p-1} \equiv 1 \pmod{p}$

- Recall $\{1a, 2a, 3a, \ldots, (p-1)a\} \equiv \{1, 2, \ldots, p-1\} \pmod{p}$
- Multiply all elements of both sides:

$$(p-1)!a^{p-1} \equiv (p-1)! \pmod{p}$$

- Canceling out $(p-1)!$ from both sides, we have the theorem!

Primality Testing

- Given a number N, we can use Fermat's theorem as a probabilistic test to see if it is prime:
  - if $a^{N-1} \not\equiv 1 \pmod{N}$ then N is not prime
  - Repeat with different values of a to gain more confidence

Primality Testing

- Given a number N, we can use Fermat's theorem as a probabilistic test to see if it is prime:
  - if $a^{N-1} \not\equiv 1 \pmod{N}$ then N is not prime
  - Repeat with different values of a to gain more confidence
- *Question:* If N is *not* prime, what is the probability that the above procedure will fail?

Primality Testing

- Given a number N, we can use Fermat's theorem as a probabilistic test to see if it is prime:
  - if $a^{N-1} \not\equiv 1 \pmod{N}$ then N is not prime
  - Repeat with different values of a to gain more confidence
- *Question:* If N is *not* prime, what is the probability that the above procedure will fail?
  - For Carmichael's numbers, the probability is 1 — but ignore this for now, since these numbers are very rare.

Primality Testing

- Given a number N, we can use Fermat's theorem as a probabilistic test to see if it is prime:
  - if $a^{N-1} \not\equiv 1 \pmod{N}$ then N is not prime
  - Repeat with different values of a to gain more confidence
- *Question:* If N is *not* prime, what is the probability that the above procedure will fail?
  - For Carmichael's numbers, the probability is 1 — but ignore this for now, since these numbers are very rare.
  - For other numbers, we can show that the above procedure works with probability 0.5

Primality Testing

Lemma

If $a^{N-1} \not\equiv 1 \pmod{N}$ *for a relatively prime to N, then it holds for at least half the choices of* $a < N$.

Primality Testing

Lemma

If $a^{N-1} \not\equiv 1 \pmod{N}$ for a relatively prime to N, then it holds for at least half the choices of $a < N$.

- If there is no b such that $b^{N-1} \equiv 1 \pmod{N}$, then we have nothing to prove.

Primality Testing

Lemma

If $a^{N-1} \not\equiv 1 \pmod{N}$ *for a relatively prime to N, then it holds for at least half the choices of* $a < N$.

- If there is no b such that $b^{N-1} \equiv 1 \pmod{N}$, then we have nothing to prove.
- Otherwise, pick one such b, and consider $c \equiv ab$.

Primality Testing

Lemma

If $a^{N-1} \not\equiv 1 \pmod{N}$ for a relatively prime to N, then it holds for at least half the choices of $a < N$.

- If there is no b such that $b^{N-1} \equiv 1 \pmod{N}$, then we have nothing to prove.
- Otherwise, pick one such b, and consider $c \equiv ab$.
- Note $c^{N-1} \equiv a^{N-1}b^{N-1} \equiv a^{N-1} \not\equiv 1$

Primality Testing

Lemma

If $a^{N-1} \not\equiv 1 \pmod N$ for a relatively prime to N, then it holds for at least half the choices of $a < N$.

- If there is no b such that $b^{N-1} \equiv 1 \pmod N$, then we have nothing to prove.
- Otherwise, pick one such b, and consider $c \equiv ab$.
- Note $c^{N-1} \equiv a^{N-1}b^{N-1} \equiv a^{N-1} \not\equiv 1$
- Thus, for every b for which Fermat's test is satisfied, there exists a c that does not satisfy it.
  - Moreover, since a is relatively prime to N, $ab \not\equiv ab'$ unless $b \equiv b'$.

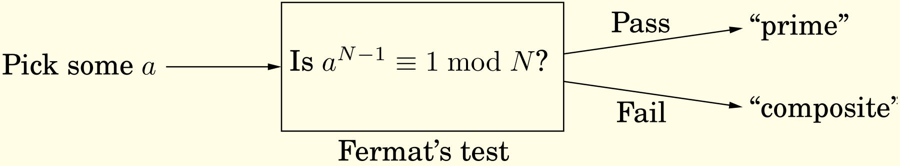
Primality Testing

Lemma

If $a^{N-1} \not\equiv 1 \pmod{N}$ for a relatively prime to N, then it holds for at least half the choices of $a < N$.

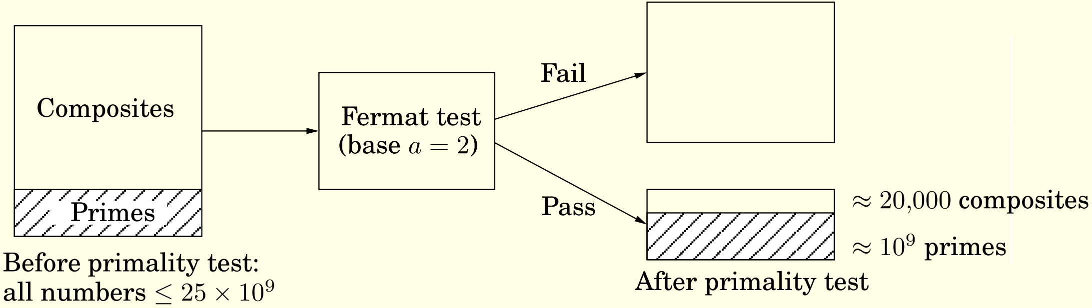
- If there is no b such that $b^{N-1} \equiv 1 \pmod{N}$, then we have nothing to prove.
- Otherwise, pick one such b, and consider $c \equiv ab$.
- Note $c^{N-1} \equiv a^{N-1}b^{N-1} \equiv a^{N-1} \not\equiv 1$
- Thus, for every b for which Fermat's test is satisfied, there exists a c that does not satisfy it.
  - Moreover, since a is relatively prime to N, $ab \not\equiv ab'$ unless $b \equiv b'$.
- Thus, at least half of the numbers $x < N$ relatively prime to N will fail the test.

Primality Testing



- When Fermat's test returns "prime" $Pr[N \text{ is not prime}] < 0.5$
- If Fermat's test is repeated for k choices of a, and returns "prime" in each case, $Pr[N \text{ is not prime}] < 0.5^k$
- In fact, 0.5 is an upper bound. Empirically, the probability has been much smaller.

Primality Testing



- Empirically, on numbers less than 25 billion, the probability of Fermat's test failing to detect non-primes (with $a = 2$) is more like 0.00002
- This probability decreases even more for larger numbers.

Prime number generation

Lagrange's Prime Number Theorem

For large N, primes occur approx. once every $\log N$ numbers.

Prime number generation

Lagrange's Prime Number Theorem

For large N, primes occur approx. once every $\log N$ numbers.

Generating Primes

- Generate a random number
- Probabilistically test it is prime, and if so output it
- Otherwise, repeat the whole process

Prime number generation

Lagrange's Prime Number Theorem

For large N, primes occur approx. once every $\log N$ numbers.

Generating Primes

- Generate a random number
- Probabilistically test it is prime, and if so output it
- Otherwise, repeat the whole process

- What is the complexity of this procedure?

Prime number generation

Lagrange's Prime Number Theorem

For large N, primes occur approx. once every $\log N$ numbers.

Generating Primes

- Generate a random number
- Probabilistically test it is prime, and if so output it
- Otherwise, repeat the whole process

- What is the complexity of this procedure?
  - $O(\log^2 N)$ multiplications on $\log N$ bit numbers
- If N is not prime, should we try $N + 1, N + 2, \ldots$ instead of generating a new random number?

Prime number generation

Lagrange's Prime Number Theorem

For large N, primes occur approx. once every $\log N$ numbers.

Generating Primes

- Generate a random number
- Probabilistically test it is prime, and if so output it
- Otherwise, repeat the whole process

- What is the complexity of this procedure?
  - $O(\log^2 N)$ multiplications on $\log N$ bit numbers
- If N is not prime, should we try $N+1, N+2, \ldots$ instead of generating a new random number?
  - No, it is not easy to decide when to give up.

Rabin-Miller Test

- Works on Carmichael's numbers
- For prime number test, we consider only odd N, so $N - 1 = 2^t u$ for some odd u
- Compute

$$a^u, a^{2u}, a^{4u}, \ldots, a^{2^t u} = a^{N-1}$$

- If a^{N-1} is not 1 then we know N is composite.
- Otherwise, we do a follow-up test on a^u, a^{2u} etc.
    - Let $a^{2^r u}$ be the first term that is equivalent to 1.
    - If $r > 0$ and $a^{2^{r-1}u} \not\equiv -1$ then N is composite
- This combined test detects non-primes with a probability of at least 0.75 for all numbers.